International Journal of Engineering and Applied Sciences (IJEAS)
ISSN: 2394-3661, Volume-3, Issue-8, August 2016

On The Existence of A Unique Solution for Systems of Ordinary Differential Equations of First Order

Abdussalam A. Bojeldain, Saif Alislam E. Muhammed

***Abstract*— In this paper, we state and prove a theorem for local existence of a unique solution for a system of nonlinear ordinary differential equations (NODE) of first order by proving that the nonlinear operator of this system is contractive in a metric space subset of the Banach space consisting of the bounded differentiable functions on the real numbers and equipped with a Bielescki's type norm. Finally, we give examples to illustrate our result.**

***Index Terms*— Banach space of bounded functions $X(t) \in C'(\mathbb{R})$, Existence of *a* unique solution globally, System of nonlinear ordinary differential equations of first order.**

I. INTRODUCTION

In 2015, Bojeldain [1] proved a theorem for the existence of a unique solution for nonlinear ordinary differential equations of order m.

In this paper we study the system of nonlinear ordinary differential equations of first order having the general form:

$$X'(t) = F(t, X(t)), \tag{1}$$

with the initial condition,

$$X(a) = C, \tag{2}$$

where $t \geq a$ is a finite real number, and

$$X'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} \tag{3}$$

$$X(a) = \begin{bmatrix} x_1(a) \\ x_2(a) \\ x_3(a) \\ \vdots \\ x_n(a) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} \tag{4}$$

$$F(t, X(t)) = \begin{bmatrix} f_1(t, X(t)) \\ f_2(t, X(t)) \\ f_3(t, X(t)) \\ \vdots \\ f_n(t, X(t)) \end{bmatrix} \tag{5}$$

In other form the system is

$$x_i'(t) = f_i(t, x_1(t), x_2(t), x_3(t), \cdots, x_n(t))$$

for $i = 1,2,3,\cdots,n$.

Since the system of (NODE) (1) with the initial condition (2) is equivalent to the integral equation:

$$X(t) = C + \int_a^t F(\tau, X(\tau))d\tau \tag{7}$$

we denote the right hand side (r.h.s.) of (7) by the nonlinear operator $Q(X)t$; then prove that this operator is contractive in a metric space E subset of the Banach space B of the class of bounded functions $X(t) \in C'(\mathbb{R})$ defined by:

$$B = \{(t, X(t)) \mid |t-a| < \infty, |x_i(t) - c_i| \leq T_i' < \infty, \ i = 1,2,3,\cdots,n\} \tag{8}$$

and equipped with the weighted norm:

$$\|X\| = \max_{|t-a|<\infty} (exp(-L|t-a|) \textstyle\sum_{i=1}^n |x_i(t)|) \tag{9}$$

which is known as Bielescki's type norm [2], $L = max(l, 1)$ is a finite real number where $l = max(l_i)$, l_i is the Lipschitz coefficient of $f_i(t, X(t))$ for $i = 1,2,3,\cdots,n$ in $B1$ (a subset of the Banach space B given by (8)) defined by:

$$B1 = \{(t, x_1(t), x_2(t), x_3(t), \cdots, x_n(t)) \mid |t-a| \leq T, |x_i(t) - c_i| \leq T_i \leq T_i'\} \tag{10}$$

where

T and T_i for $i = 1,2,\dots,n$ are finite real numbers.

When the function F in the r.h.s of (1) depends linearly on its arguments except t, then equation (1) is a 1^{st} order system of linear ordinary differential equations and to prove the existence of a unique solution for it in $[a-T, a+T]$ one usually prove that component wise in a neighbourhood $N_\delta(a)$ for $t \in [a, a+\delta]$, then mimic the same steps of the proof for $t \in [a-\delta, a]$; after that use another theorem to show whether the solution do exist for all $t \in [a-T, a+T]$ or not as in [3]. By the theorem which we are going to state and prove in this paper one can easily prove the existence of a unique solution for 1^{st} order nonlinear systems of ordinary differential equations on the general form (1) for all $t \in [a-\delta, a+\delta]$ directly in a very simple metric space $\mathbb{E}$ consisting of the functions $X(t) \in C'[a-T, a+T]$, subset of the Banach space (8) [4], and equipped with the simple efficient norm (9) for $|t-a| \leq \delta$, moreover if the Lipschitz condition (11) is guaranteed to be satisfied in the Banach space (8), then the theorem guarantees the existence of a unique solution for $|t-a| < \infty$ in most cases and not in general as mentioned in [5].

Abdussalam A. Bojeldain, Department of Mathematics, Omar Al-Mukhtar University, El-Beida, Libya, 00218 7464804.
Saif Alislam E. Muhammed, Department of Mathematics, Omar Al-Mukhtar University, El-Beida, Libya, 00218 927556629.

Note that this theorem is valid for 1^{st} order linear systems of ordinary differential equations as well.

II. MAIN THEOREM

Now, we are in a position to state and prove our main theorem.

Theorem: Consider the system of (NODE) (1) with the initial condition (2) and suppose that the function F in the r.h.s. of (1) is continuous and satisfies the Lipschitz condition:

$$|F(t,X(t))-F(t,Y(t))| \leq l|X(t)-Y(t)| = = l\sum_{i=1}^{n}|x_i - y_i| \quad (11)$$

in $B1$ given by (10); then the initial value problem (1) and (2) has a unique solution in the $(n+1)$ dimensional metric space E (of the functions $X(t) \in C'[a-\delta, a+\delta]) \subseteq B$ defined by:

$$E = \{(t, x_1(t), x_2(t), x_3(t), \cdots, x_n(t)) | \, |t-a| \leq \delta, |x_i(t) - c_i| \leq T^*\} \quad (12)$$

such that $\delta = min\,(T, \frac{T^*}{M})$; where $T^* = min(T_i)$, $M = max(M_i)$, and $|f_i(t, X(t))| \leq M_i$, $for\ i = 1,2,3, \cdots, n\ in\ B1$
.

Proof: Integrating both sides of (1) from a to t and using the initial condition(2), we obtain the system of integral equations(7).

To form a fixed point problem $X(t) = Q(X)t$ denote the r.h.s. of (7) by $Q(X)t$, and to apply the contraction mapping theorem we first show that $Q: E \to E$; then prove that Q is contractive in E.

We see that:

$$. \quad |Q(X)t - C| = \left|\int_a^t F(\tau, X(\tau))d\tau\right| \leq$$
$$\leq \int_a^t |F(\tau, X(\tau))|d\tau \leq$$
$$\leq \int_a^t M\, d\tau \leq M|t-a| \leq M\delta \leq M\frac{T^*}{M} \leq T^* \quad (13)$$

hich means that $Q: E \to E$.

Next we prove that Q is contractive, to do so we consider the difference:

$$|Q(X)t - Q(Y)t| = |Q(X) - Q(Y)|(t) =$$
$$= \left|\int_a^t \left(F(\tau, X(\tau)) - F(\tau, Y(\tau))\right) d\tau\right| \leq$$
$$\leq \int_a^t |F(\tau, X(\tau)) - F(\tau, Y(\tau))| d\tau \quad (14)$$

which according to Lipschitz condition (11) yields:

$$|Q(X) - Q(Y)|(t) \leq l\int_a^t |X(\tau) - Y(\tau)|d\tau \leq$$
$$\leq L\int_a^t \sum_{i=1}^{n}|x_i(\tau) - y_i(\tau)|d\tau \quad (15)$$

Multiplying the most r. h. s. of (15) by $exp(-L|\tau - a|)\, exp(L|\tau - a|)$, we get

$$|Q(X) - Q(Y)|(t) \leq$$
$$L\int_a^t \left(\sum_{i=1}^{n} |x_i(\tau) - y_i(\tau)|\, exp(-L|\tau - a|)\right) exp(L|\tau - a|)\, d\tau \leq L\int_a^t \max_{|t-a|\leq\delta}\left(exp(-L|\tau - a|) \sum_{i=1}^{n} |x_i(\tau) - y_i(\tau)|\right) exp(L|\tau - a|)\, d\tau \quad (16)$$

which is (according to (9)),

$$|Q(X) - Q(Y)|(t) \leq L\|X - Y\| \int_a^t exp(L|\tau - a|)\, d\tau =$$
$$= \|X - Y\|(exp(L|t-a|) - 1) \quad (17)$$

i.e.,

$$|Q(X) - Q(Y)|(t) \leq \|X - Y\|(exp(L|t-a|) - 1)$$
(18)

Multiplying both sides of (18) by $exp(-L|t-a|)$ leads to:

$$exp(-L|t-a|)|Q(X) - Q(Y)|(t) \leq \|X - Y\| \cdot (1 - exp(-L|t-a|)) \leq \|X - Y\|(1 - exp(-L\delta)) \quad (19)$$

The most r. h. s. of (19) is independent of t, thus it is an upper bound for its l. h. s. for any $|t - a| \leq \delta$; whence:

$$\max_{|t-a|\leq\delta}\left(exp(-L|t-a|)|Q(X) - Q(Y)|(t)\right) \leq$$
$$\leq \|X - Y\|(1 - exp(-L\delta)) \quad (20)$$

which, according to the norm definition (9), gives:

$$\|Q(X) - Q(Y)\| \leq (1 - exp(-L\delta))\|X - Y\| \quad (21)$$

Since $0 < (1 - exp(-L\delta)) < 1$; then $Q(X)t$ is a contraction operator in E and has a unique solution for $t \in N_\delta(a)$.

III. EXAMPLS

In this section, we give two examples illustrate the above obtained result.

Example 3.1 We selected the exact solutions:

$$\left.\begin{array}{r} x_1^*(t) = t^2 \\ x_2^*(t) = e^t \\ x_3^*(t) = t + 4 \end{array}\right\}, \quad (22)$$

and constructed the following system of nonlinear ordinary differential equations:

$$\left.\begin{array}{r} x_1'(t) = 2t - x_1^2 + 2t^2x_1 - t^4 \\ x_2'(t) = 2e^t - x_2 \\ x_3'(t) = -x_3^2 + 2x_3t + 8x_3 - t^2 - 8t - 15 \end{array}\right\} \quad (23)$$

If we a= 0 in (22), we get

$$\left.\begin{array}{r} x_1^*(0) = 0 \\ x_2^*(0) = 1 \\ x_3^*(0) = 4 \end{array}\right\}, \quad (24)$$

as the initial conditions to (23).

Selecting positive finite real numbers $T_1, T_2, and\ T_3$ we find that $|x_i - c_i| \leq T_i$ leads to $|x_1(t)| \leq T_1, |x_2(t)| \leq T_2 + 1, |x_3(t)| \leq T_3 + 4.$

The subset $B1$ is:

$\{(t,x_1(t),x_2(t)) \mid |t-a| \le T, |x_1(t)| \le T_1, \ |x_2(t)| \le T_2 + 1, |x_3(t)| \le T_3 + 4 \}$ (25)

In $B1$ we have:

$|f_1(t,x_1(t),x_2(t),x_3(t))| = |2t - x_1^2 + 2t^2 x_1 - t^4| \le$

$\le 2T + T_1^2 + 2T^2T_1^2 + T^4$,

$|f_2(t,x_1(t),x_2(t),x_3(t))| = |2e^t - x_2| \le 2e^T + T_2 + 1$,

and

$|f_3(t,x_1(t),x_2(t),x_3(t))| =$

$= |-x_3^2 + 2x_3 t + 8x_3 - t^2 - 8t - 15| \le$

$\le (T_3+4)(T_3+2T+12) + T(T+8) + 15$

i.e. $M_1 = 2T + T_1^2 + 2T^2T_1^2 + T^4$, $M_2 = 2e^T + T_2 + 1$, $and\ M_3 = (T_3+4)(T_3+2T+12) + T(T+8) + 15$.

Next, we check the Lipschitz condition for $f_1, f_2, and\ f_3$:

$|f_1(t,X(t) - f_1(t,Y(t))| = |-x_1^2 + 2t^2x_1 + y_1^2 - 2t^2y_1| \le$

$\le 2(T_1 + T^2)(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|)$ (26)

therefore f_1 satisfies the Lipschitz condition (11) in $B1$ given by (25) with Lipschitz coefficient $l_1 = 2(T_1 + T^2)$,

$|f_2(t,X(t)) - f_2(t,Y(t))| = |2e^t - x_2 - 2e^t - y_2| \le$

$\le |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|$, (27)

i.e. f_2 satisfies the Lipschitz condition (11) in $B1$ given by (25) with Lipschitz coefficient $l_2 = 1$,

and

$|f_3(t,X(t)) - f_3(t,Y(t))| =$

$= |-x_3^2 + 2x_3t + 8x_3 + y_3^2 - 2y_3t - 8y_3| \le$

$\le 2(8 + T_3 + T)(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|)$; (28)

whence f_3 satisfies the Lipschitz condition (11) in $B1$ given by (25) with Lipschitz coefficient $l_3 = 2(8 + T_3 + T)$.

Therefore $L = max(2(T_1 + T^2), 1, 2(8 + T_3 + T))$.

Putting $M = max(M_1, M_2, M_3) = k_2 T\ and\ T^* = min(T_1, T_2, T_3) = k_1 T$ such that the k_1, k_2 are positive real numbers, we find that the unique solution exists in the interval $|t - a| \le \delta$ where,

$$\delta = \begin{cases} T, & if\ k_1 \le k_2 \\ \frac{k_1}{k_2} T, & if\ k_2 > k_1 \end{cases}.$$

Example 3.2 [5] As a second example, consider the following system of nonlinear ordinary differential equations:

$$\left. \begin{array}{r} x_1' + t^2 \cos x_1 + x_2 = 0 \\ x_2' + \sin x_1 = 0 \end{array} \right\}, \qquad (29)$$

having the initial conditions:

$$\left. \begin{array}{r} x_1(a) = 0 \\ x_2(a) = 1 \end{array} \right\}. \qquad (30)$$

Selecting positive finite real numbers T_1, T_2 we find that $|x_i - c_i| \le T_i$ leads to $|x_1(t)| \le T_1$, $|x_2(t)| \le T_2 + 1$.

The subset $B1$ is:

$\{(t,x_1(t),x_2(t)) \mid |t-a| \le T, |x_1(t)| \le T_1, \ |x_2(t)| \le T_2 + 1\}$ (31)

in which,

$|f_1(t,x_1(t),x_2(t))| = |-t^2 \cos x_1 - x_2| \le t^2(|\cos x_1|) + |x_2| \le (T+a)^2 + T_2 + 1$

and $|f_2(t,x_1(t),x_2(t))| = |-\sin x_1| \le 1$,

i.e. $M_1 = (T+a)^2 + T_2 + 1$, $M_2 = 1$. Hence $M = max(M_1, M_2) = (T+a)^2 + T_2 + 1$.

Next, we check the Lipschitz condition for f_1 and f_2:

$|f_1(t,X(t)) - f_1(t,Y(t))| =$

$= |-t^2 \cos x_1 - x_2 + t^2 \cos y_1 + y_2| \le$

$\le t^2 |\cos x_1 - \cos y_1| + |x_2 - y_2| \le$

$\le 2(T+a)^2(|x_1 - y_1| + |x_2 - y_2|)$, (32)

and

$|f_2(t,X(t)) - f_2(t,Y(t))| = |-\sin x_1 + \sin y_1| \le$

$\le |x_1 - y_1| \le (|x_1 - y_1| + |x_2 - y_2|)$ (33)

therefore f_1 and f_2 satisfie the Lipschitz condition (11) in $B1$ given by (31) with Lipschitz coefficient $l_1 = 2(T+a)^2$ and $l_2 = 1$ respectively.

Hence $L = max(l, 1) = 2(T+a)^2$.

Putting $M = k_2 T$, $T^* = k_1 T$ such that the k_1, k_2 are positive real numbers, we find that the unique solution exists in the interval $|t - a| \le \delta$ where,

$$\delta = \begin{cases} T, & if\ k_1 \le k_2 \\ \frac{k_1}{k_2} T, & if\ k_2 > k_1 \end{cases}$$

IV. CONCLUSION

We see that the contraction coefficient $0 < (1 - exp(-L\delta)) < 1$ for any finite $\delta > 0$. Moreover, in most cases, if the function F in the r.h.s. of (1) is continuous and satisfies Lipschitz condition in the Banach space (8) with finite positive Lipschitz coefficient, then the theorem is proved for t in any interval I of finite length because the contraction coefficient $(1 - exp(-L\mu(I)))$ will be positive and less than 1; where $\mu(I)$ is the measure of the interval I.

REFERENCES

[1] A. A. Bojeldain, On The Existence of a Unique Solution for Nonlinear Ordinary Differential Equations of Order m,‘‘ vol. 30, no. 1, pp. 10-17, 2015.

[2] A. Bielescki, Ramarks on the applications of the Banach Kantorowich-Tichonoff method for the equation $S = f(x, y, z, p, q)$,‘‘ Acad. Polon. Bul. Sci. vol. IV, no. 5, pp. 259-262, 1956.

[3] W. Hurewicz, Lectures On Ordinary Differential Equations,‘‘ The M. I. T. Press, 1974.

[4] V. Hutson and J. S. Pym, Application of Functional Analysis and Operator Theory,‘‘ Academic press, 1980.

[5] B. Jankő, Nomerical Methods for Solving Nonlinear Operator Equations,‘‘ Eőtvős Loránd University Puplising House, Budapest, 1990.

Abdussalam A. Bojeldain, Department of Mathematics, Omar Al-Mukhtar University, El-Beida, Libya, 00218 7464804, P. O. Box 919 El-Beida, Libya.

Saif Alislam E. Muhammed, Department of Mathematics, Omar Al-Mukhtar University, El-Beida, Libya, 00218 927556629, P. O. Box 919 El-Beida, Libya.