Applications of Two Dimensional Fractional Fourier-Mellin Transform to Differential Equations

V.D. Sharma, P.B. Deshmukh

Abstract— **Integral transforms play wide and important role in mathematical physics, theoretical physics. Fourier-Mellin transform is used in fields like electronics, agriculture, medical etc.**

It has applications as registration of images, watermarks, invariant pattern recognition, preprocessing of images.

In this paper we have obtained the differential operator ∧ and ∧*. Using it we have solved the differential equation of the type . Also an application of two-dimensional fractional Fourier-Mellin transform to differential equation is presented.

Index Terms— **Two-Dimensional Fractional Fourier-Mellin Transform, Testing Function Space, Generalized function.**

I. INTRODUCTION

Invariant content-based image retrieval using a complete set of Fourier-Mellin descriptors works better for database containing isolated objects on a uniform background, it seems well-suited for professional database such as : medical, biology, telecommunication etc [1].

Hyperspectral imagery is playing important role in many fields such as geology, agriculture, environment, military, atmosphere and so on. Image registration with hyperspectral data based on fourier-mellin transform [2]. A visual odometry method that estimates the location and orientation of a robotic rover platform. Visual Odometry Based on the Fourier-Mellin Transform for a Rover Using a Monocular Ground-Facing Camera [3]. RMI, RTMT are basically used in medical for image registration is based on fourier-mellin transform . This transform gives a fast registration and good registration accuracy and avoid trapping in the local optima [4].

Fourier-Mellin transform provides a global method for registering images in a video sequence from which the rotation and translation of the camera motion can be estimated. It has been ability of the gray level image representation for pattern recognition. FMT is used to identify plant leaves at various life stages based on the leaves shape or contour [9]. Milanese, R.,Cherbuliez, M.,Pun, T. was looking for the invariant content of images in multimedia archives using Fourier-Mellin transformation[10].

Motivated by the above work, in our previous work we have generalized two dimensional fractional Fourier-Mellin transform in the distributional sense. Also proved the analyticity theorem, an inversion theorem, convolution theorem, also calculated adjoint operators. [5,6,7,8]. In the present work we have obtained the differential operator ∧ and ∧*. Using it we have solved the differential equation of the type $P(\wedge^*_{x,y,t,q})v = g$. Also an application of two-dimensional fractional Fourier-Mellin transform to differential equation is calculated.

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II. TWO-DIMENSIONAL FRACTIONAL FOURIER-MELLIN TRANSFORM

A. *Definition of two-dimensional fractional Fourier-Mellin transform:*

The two-dimensional fractional Fourier-Mellin transform with parameters α and θ of $f(x,y,t,q)$ denoted by 2DFRFMT$\{f(x,y,t,q)\}$ performs a linear operation, given by the integral transform. 2DFRFMT

$$\{f(x,y,t,q)\} = F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} f(x,y,t,q)K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\,dxdydtdq \quad \text{----(1)}$$

Where $K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)$

$$= \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{1}{2sin\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)]}$$

$$t^{\frac{2\pi i\lambda}{sin\theta}-1} q^{\frac{2\pi i\chi}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[\lambda^2+\chi^2+log^2 t+log^2 q]}$$

$$= C_{1\alpha} e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)]}$$

$$t^{C_{1\theta}i\lambda-1} q^{C_{1\theta}i\chi-1} e^{C_{2\theta}i[\lambda^2+\chi^2+log^2 t+log^2 q]}$$

where $C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}$, $C_{2\alpha} = \frac{1}{2sin\alpha}$ $C_{1\theta} = \frac{2\pi}{sin\theta}$,

$C_{2\theta} = \frac{\pi}{tan\theta}$ $\quad 0 < \alpha < \frac{\pi}{2},\ 0 < \theta < \frac{\pi}{2}$ ---(2)

B. *The Test Function*

An infinitely differentiable complex valued smooth function $\emptyset(x,y,t,q)$ on R^n belongs to $E(R^n)$, if for each compact set $I \subset S_{a,b}$, $J \subset S_{c,d}$ where

$S_{a,b} = \{x,y : x,y\epsilon R^n, |x| \le a, |y| \le b, a > 0, b > 0\}$

$S_{c,d} = \{t,q : t,q\epsilon R^n, |t| \le c, |q| \le d, c > 0, d > 0\}$

$\gamma_{E,m,n,k,l}[\emptyset(x,y,t,q)] = \sup_{\substack{x,y\epsilon I\\ t,q\epsilon J}} \left|D^{m,n,k,l}_{x,y,t,q}\emptyset(x,y,t,q)\right| < \infty$ ---(3)

Thus $E(R^n)$ will denote the space of all $\emptyset(x,y,t,q)\epsilon E(R^n)$ with compact support contained in

$S_{a,b} \cap S_{c,d}$.

Note that the space E is complete and therefore a Frechet space. Moreover, we say that $f(x,y,t,q)$ is a fractional Fourier-Mellin transformable if it is a member of E.

III. Distributional Two Dimensional Fractional Fourier-Mellin Transform (2DFRFMT)

The two dimensional distributional Fractional Fourier -Mellin transform of $f(x,y,t,q)\epsilon E^*(R^n)$ can be defined by

$$2DFRFMT\{f(x,y,t,q)\} = F_{\alpha,\theta}(\xi,\eta,\lambda,\chi) = \langle f(x,y,t,q), K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle \quad \text{---(4)}$$

where, $K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)$

$$= \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{1}{2sin\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}\, t^{\frac{2\pi i\lambda}{sin\theta}-1} q^{\frac{2\pi i\chi}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[\lambda^2+\chi^2+log^2t+log^2q]}$$

$$= C_{1\alpha}e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}\, t^{C_{1\theta}i\lambda-1} q^{C_{1\theta}i\chi-1} e^{C_{2\theta}i[\lambda^2+\chi^2+log^2t+log^2q]}$$

where $C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}$, $C_{2\alpha} = \frac{1}{2sin\alpha}$, $C_{1\theta} = \frac{2\pi}{sin\theta}$,

$C_{2\theta} = \frac{\pi}{tan\theta}$, $0 < \alpha < \frac{\pi}{2}$, $0 < \theta < \frac{\pi}{2}$ ---(5)

Right hand side of equation (4) has a meaning as the application of $f(x,y,t,q)\epsilon E^*(R^n)$ to $K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\epsilon E$.

It can be extended to the complex space as an entire function given by

$$2DFRFMT\{f(x,y,t,q)\} = F_{\alpha,\theta}(\xi',\eta',\lambda',\chi')$$

$$= \langle f(x,y,t,q), K_{\alpha,\theta}(x,y,t,q,\xi',\eta',\lambda',\chi')\rangle \quad \text{---(6)}$$

The right hand side is meaningful because for each $\xi',\eta',\lambda',\chi'\epsilon C^n$, $K_{\alpha,\theta}(x,y,t,q,\xi',\eta',\lambda',\chi')\epsilon E$ as a function of x,y,t,q.

IV. Application of Two Dimensional Fractional Fourier-Mellin Transform

The kernel of two Dimensional Fractional Fourier-Mellin transform as

$$K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi) = C_{1\alpha}e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}\, t^{C_{1\theta}i\lambda-1} q^{C_{1\theta}i\chi-1} e^{C_{2\theta}i[\lambda^2+\chi^2+log^2t+log^2q]}$$

$$D_xD_yD_tD_q[K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)]$$

$$= C_{1\alpha}e^{iC_{2\alpha}(\xi^2+\eta^2)cos\alpha} D_x[e^{iC_{2\alpha}[x^2cos\alpha-2(x\xi)]}]\, D_y[e^{iC_{2\alpha}[y^2cos\alpha-2(y\eta)]}]\, e^{C_{2\theta}i[\lambda^2+\chi^2]} D_t[t^{C_{1\theta}i\lambda-1}e^{C_{2\theta}ilog^2t}]\, D_q[q^{C_{1\theta}i\chi-1}e^{C_{2\theta}ilog^2q}]$$

$$= \{C_{1\alpha}e^{iC_{2\alpha}(\xi^2+\eta^2)cos\alpha}\, e^{iC_{2\alpha}[x^2cos\alpha-2(x\xi)]} iC_{2\alpha}[2xcos\alpha-2\xi]\, e^{iC_{2\alpha}[y^2cos\alpha-2(y\eta)]} iC_{2\alpha}[2ycos\alpha-2\eta]\}$$
$$\{[e^{iC_{2\theta}[\lambda^2+\chi^2]}t^{C_{1\theta}i\lambda-1}e^{C_{2\theta}ilog^2t}(iC_{2\theta})\frac{2logt}{t} + e^{C_{2\theta}ilog^2t}(C_{1\theta}i\lambda-1)t^{C_{1\theta}i\lambda-2}]\, [q^{C_{1\theta}i\chi-1}e^{C_{2\theta}ilog^2q}(iC_{2\theta})\frac{2logq}{q} + e^{C_{2\theta}ilog^2q}(C_{1\theta}i\chi-1)q^{C_{1\theta}i\chi-2}]\}$$

$$= -4C_{2\alpha}^2(xcos\alpha-\xi)(ycos\alpha-\eta)\, C_{1\alpha}e^{iC_{2\alpha}(\xi^2+\eta^2)cos\alpha}e^{iC_{2\alpha}[(x^2cos\alpha+y^2cos\alpha)-2(x\xi+y\eta)]}\, e^{iC_{2\theta}(\lambda^2+\chi^2)}t^{C_{1\theta}i\lambda-1}q^{C_{1\theta}i\chi-1}e^{iC_{2\theta}(log^2t+log^2q)}$$
$$[(iC_{2\theta})\frac{2logt}{t} + (C_{1\theta}i\lambda-1)t^{-1}]\, [(iC_{2\theta})\frac{2logq}{q} + (C_{1\theta}i\chi-1)q^{-1}]$$

$$= -4C_{2\alpha}^2(xcos\alpha-\xi)(ycos\alpha-\eta)\frac{1}{tq}(2iC_{2\theta}logt+C_{1\theta}i\lambda-1)(2iC_{2\theta}logq+C_{1\theta}i\chi-1)\, C_{1\alpha}e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)]}t^{C_{1\theta}i\lambda-1}q^{C_{1\theta}i\chi-1}e^{iC_{2\theta}(\lambda^2+\chi^2+log^2t+log^2q)}$$

$$= -4\frac{1}{4sin^2\alpha}(xcos\alpha-\xi)(ycos\alpha-\eta)\frac{1}{tq}\, 2i\left(C_{2\theta}logt+C_{1\theta}i\lambda-\frac{i}{2}\right)2i\left(C_{2\theta}logq+C_{1\theta}i\chi-\frac{i}{2}\right)K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)$$

$$D_xD_yD_tD_q[K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)]$$
$$= (4cosec^2\alpha)(xcos\alpha-\xi)(ycos\alpha-\eta)\frac{1}{tq}\left(C_{2\theta}logt+C_{1\theta}i\lambda-\frac{i}{2}\right)\left(C_{2\theta}logq+C_{1\theta}i\chi-\frac{i}{2}\right)K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)$$

$$x^{-1}y^{-1}tqD_xD_yD_tD_q[K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)]$$
$$= (4cosec^2\alpha)xy\left(cos\alpha-\frac{\xi}{x}\right)\left(cos\alpha-\frac{\eta}{y}\right)\frac{1}{tq}\left(tC_{2\theta}logt+tC_{1\theta}i\lambda-\frac{it}{2}\right)\left(qC_{2\theta}logq+qC_{1\theta}i\chi-\frac{iq}{2}\right)K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)$$

$$\wedge_{x,y,t,q}K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)$$
$$= x^{-1}y^{-1}tqD_xD_yD_tD_q[K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)] - (4cosec^2\alpha)xy\left(cos\alpha-\frac{\xi}{x}\right)\left(cos\alpha-\frac{\eta}{y}\right)\frac{1}{tq}\left(tC_{2\theta}logt+tC_{1\theta}i\lambda-\frac{it}{2}\right)\left(qC_{2\theta}logq+qC_{1\theta}i\chi-\frac{iq}{2}\right)K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)$$

$$\wedge_{x,y,t,q} = x^{-1}y^{-1}tqD_xD_yD_tD_q - (4cosec^2\alpha)xy\left(cos\alpha-\frac{\xi}{x}\right)\left(cos\alpha-\frac{\eta}{y}\right)\frac{1}{tq}\left(tC_{2\theta}logt+tC_{1\theta}i\lambda-\frac{it}{2}\right)\left(qC_{2\theta}logq+qC_{1\theta}i\chi-\frac{iq}{2}\right)$$

$$\wedge_{x,y,t,q}K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi) = (4cosec^2\alpha)K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)$$

$$\wedge_{x,y,t,q} = (4cosec^2\alpha)$$

$$\wedge_{x,y,t,q}K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi) = (4cosec^2\alpha)K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)$$

$$\wedge^2_{x,y,t,q}K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi) = (4cosec^2\alpha)^2K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)$$

$$\wedge^3_{x,y,t,q}K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi) = (4cosec^2\alpha)^3K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)$$

$\therefore \wedge^{k}{}_{x,y,t,q}\, K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi) = (4cosec^2\alpha)^k$
$K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)$

Since the operator

$\therefore \wedge^{k}{}_{x,y,t,q}\, K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi) = (4cosec^2\alpha)^k$
$K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)$

Is obvious linear and continuous for $\alpha > 0, \theta > 0$

We have

$2DFRFMT\{\wedge^{k}{}_{x,y,t,q}\, f(x,y,t,q)\}$
$= \langle \wedge^{k}{}_{x,y,t,q}\, f(x,y,t,q), K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$
$= \langle f(x,y,t,q), \wedge^{k}{}_{x,y,t,q}\, K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$
$= \langle f(x,y,t,q), (4cosec^2\alpha)^k K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$

For all $f \in E'$ for $0 < \alpha < \frac{\pi}{2},\ 0 < \theta < \frac{\pi}{2}$.

Adjoint Operator $\wedge^{*}{}_{x,y,t,q}$

We define an operator $\wedge^{*}{}_{x,y,t,q}: E' \to E$ using the relation

$\langle \wedge^{*}{}_{x,y,t,q}\, [f(x,y,t,q)], \emptyset(x,y,t,q)\rangle$
$= \langle [f(x,y,t,q)], \wedge_{x,y,t,q}\, [\emptyset(x,y,t,q)]\rangle$

For all $f \in E'$ and $\emptyset \in E$. The operator $\wedge^{*}{}_{x,y,t,q}$ is called the adjoint operator of $\wedge_{x,y,t,q}$ for each

$k = 1,2,3,4 - - - -$

We easily get,

$\langle (\wedge^{*}{}_{x,y,t,q})^k f(x,y,t,q), K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$
$= \langle f(x,y,t,q), (\wedge^{*}{}_{x,y,t,q})^k K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$
$= \langle f(x,y,t,q), (4cosec^2\alpha)^k K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$
$= (4cosec^2\alpha)^k \langle f(x,y,t,q), K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$
$2DFRFMT\{(\wedge^{*}{}_{x,y,t,q})^k f(x,y,t,q)\} = (4cosec^2\alpha)^k$
$\{2DFRFMT\ [f(x,y,t,q)]\}$
(ξ,η,λ,χ) ---(4.1)

V. An Application of the Two Dimensional Fractional Fourier-Mellin Transform to

Differential Equations

Solution of $P(\wedge^{*}{}_{x,y,t,q})v = g$
----(5.1)

Consider, the differential equation $P(\wedge^{*}{}_{x,y,t,q})v = g$

Where, $g \in E'$ and P is any polynomial of degree m.

Suppose that the equation (5.1) possesses a solution of v.

Applying $2DFRFMT$ to equation (5.1)

We have

$2DFRFMT\{[P(\wedge^{*}{}_{x,y,t,q})]v\} = 2DFRFMT\{g\}$

if $2DFRFMT\{g\} = g^\wedge$
------(5.2)

$\therefore$ we can write

$P(4cosec^2\alpha) 2DFRFMT(v) = 2DFRFMT(g)$

$P(4cosec^2\alpha)v^\wedge = g^\wedge$ by using (5.2)
------(5.3)

If we further assume that the polynomial P is such that for $\epsilon > 0$.

$\therefore |P(4cosec^2\alpha)| < \epsilon < 0, \qquad 0 < \alpha < \frac{\pi}{2}$

Then under this assumption (5.3) gives

$v^\wedge = [\,P(4cosec^2\alpha)]^{-1}\, g^\wedge$
------(5.4)

$2DFRFMT(v) = g^\wedge[\,P(4cosec^2\alpha)]^{-1}$

Applying inversion of two dimensional fractional Fouier-Mellin transform to above equation we have

$$v = [\,2DFRFMT\,]^{-1}\left[\frac{g^\wedge}{P(4cosec^2\alpha)}\right]$$

Hence proved.

VI. Solution of Differential Equation for Two Dimensional Fractional Fourier-Mellin Transform

Consider the differential equation

$P(D)v = f(x,y,t,q)$-----------------------(6.1)

Where $f \in E'$ and $P(D) = \sum_{|\beta| \le m} a_\beta D^\beta$

is a linear differential operator of order m with constant coefficients.

Suppose that the equation (6.1) possesses a solution v.

Applying the two dimensional fractional Fourier-Mellin transform to (6.1) and using

$D_x^m D_t^n K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)$
$= \sum_{r=0}^{m}\sum_{s=0}^{n} C_{m,n} C_\alpha [xcos\alpha - \xi]^{m-2r}\,[ycos\alpha - \eta]^{n-2s}$
$e^{(m-r)u+(n-s)v}$

$$\sum_{k=0}^{l}\sum_{h=0}^{j} l!\,j! \left(\frac{2\pi i}{tan\theta}\right)^{l-k+j-h} C_k(t)\,C_k(q)(logt)^{l-k}(logq)^{h-j} e^{(l-k)w+(j-h)z}$$
$$\left(\frac{1}{t}\right)^{2l-k}\left(\frac{1}{q}\right)^{2h-j} P(\lambda)P(\chi)\, t^{\frac{2\pi i\lambda}{sin\theta}-1} q^{\frac{2\pi i\chi}{sin\theta}-1}$$

where, $C_{m,n} = \frac{m!}{r!(m-2r)!}\frac{n!}{s!(n-2s)!}(2i)^{m-r+n-s}$

$C_\alpha = cos\alpha^{r+s}(C_{2\alpha})^{m-r+n-s}$

$u = iC_{2\alpha}\{[(x)^2 + \xi^2]cos\alpha - 2(x)\xi\}$

$v = iC_{2\alpha}\{[(y)^2 + \eta^2]cos\alpha - 2(y)\eta\}$

$C_k(t) = \frac{l!}{(l-2k)!}\left[\frac{1-logt}{2tlogt}\right]^k,$

$C_k(q) = \frac{j!}{(j-2h)!}\left[\frac{1-logq}{2qlogq}\right]^h$

We have, $2DFRFMT[P(D)v] = 2DFRFMT[f] = f^\wedge$ (say)

We can reform them to the two dimensional fractional Fourier-Mellin transform and hence we get

$P(x,y,t,q)v^\wedge = f^\wedge$

Where $P(x,y,t,q)$ is polynomial in *x,y,t,q* ,

$v^\wedge = 2DFRFMT[v]$

Under the assumption that the polynomial P is such that

$P\{D_x^m D_t^n K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\} > l > 0$ for

$l_1, l_2, l_3, l_4 - - - - l_n \epsilon R^n$

$v^\wedge = P\{D_x^m D_t^n K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\} f^\wedge$

Applying inversion of two dimensional fractional Fourier-Mellin transform to above equation we have

$$v = [2DFRFMT]^{-1}\left[\frac{f^\wedge}{P\{D_x^m D_t^n K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\}}\right].$$

VII. Conclusion

We have obtained the differential operator $\wedge$ and $\wedge^*$. Using it we have solved the differential equation of the type $P(\wedge^{*}{}_{x,y,t,q})v = g$. Also an application of two-dimensional fractional Fourier-Mellin transform to differential equation is presented.

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