International Journal of Engineering and Applied Sciences (IJEAS)
ISSN: 2394-3661, Volume-3, Issue-7, July 2016

On The Dynamics Of The Difference Equation $x_{n+1}=\dfrac{\alpha x_{n-3}}{\beta+\gamma x_n x_{n-1} x_{n-2} x_{n-3}}$

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Abstract— **In this paper, we studied the global behavior of the difference equation $x_{n+1}=\dfrac{\alpha x_{n-3}}{\beta+\gamma x_n x_{n-1} x_{n-2} x_{n-3}}$ with non-negative parameters and the initial conditions x_{-3},x_{-2},x_{-1},x_0 are non-negative real numbers.**

Index Terms **— Difference Equation, Global Asymptotic Stability, Oscillation.**

I. Introduction

Difference equations are encountered in nature as the numerical solution of differential equations and difference equations have applications in population, physics, biology etc.

Although difference equations seem simple, to understand global stability of their solutions is quite difficult. One can see Refs. [1]–[6].

Cinar et al. [7], [8] investigated the global asymptotic stability of the nonlinear difference equation

$$x_{n+1}=\frac{x_{n-1}}{1+x_n x_{n-1}},n=0,1,...$$

and

$$x_{n+1}=\frac{ax_{n-1}}{1+bx_n x_{n-1}},n=0,1,...$$

Abo-Zeid et al. [9] examined the global stability and periodic character of all solutions of the difference equation

$$x_{n+1}=\frac{Ax_{n-2}}{B+Cx_n x_{n-1} x_{n-2}},n=0,1,...$$

Consider the difference equation

$$x_{n+1}=\frac{\alpha x_{n-3}}{\beta+\gamma x_n x_{n-1} x_{n-2} x_{n-3}} \qquad (0.1)$$

where the parameters α,β and γ are positive real numbers and the initial conditions x_{-3},x_{-2},x_{-1} and x_0 are real numbers. We investigate the global asymptotic behavior and the periodic character of the solutions of (0.1).

Definition 1. Let I be an interval of the real numbers and $f:I^4\to I$ be a continuously differentiable function. Consider the difference equation

$$x_{n+1}=f(x_n,x_{n-1},x_{n-2},x_{n-3}) \qquad (0.2)$$

with $x_{-3},x_{-2},x_{-1},x_0\in I$. Let $\bar{x}$ be the equilibrium point of (0.2). The linearized equation of (0.2) about the equilibrium point $\bar{x}$ is

$$y_{n+1}=c_1 y_n+c_2 y_{n-1}+c_3 y_{n-2}+c_4 y_{n-3},n=0,1,... \qquad (0.3)$$

where

$$c_1=\frac{\partial f}{\partial x_n}(\bar{x},\bar{x},\bar{x},\bar{x})$$

$$c_2=\frac{\partial f}{\partial x_{n-1}}(\bar{x},\bar{x},\bar{x},\bar{x})$$

$$c_3=\frac{\partial f}{\partial x_{n-2}}(\bar{x},\bar{x},\bar{x},\bar{x})$$

$$c_4=\frac{\partial f}{\partial x_{n-3}}(\bar{x},\bar{x},\bar{x},\bar{x})$$

The characteristic equation of (0.3) is

$$\lambda^4-\lambda^3 c_1-\lambda^2 c_2-\lambda c_3-c_4=0 \qquad (0.4)$$

Definition 2. A sequence $\{x_n\}_{n=-3}^{\infty}$ is said to be periodic with period p if $x_{n+p}=x_n$ for all $n\geq -3$.

Theorem 1. Assume that $p_i\in\mathbb{R},i=1,2,...$. Then

$$\sum_{i=1}^{4}|p_i|<1,$$

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is a sufficient condition for the asymptotic stability of (0.3).

II. DYNAMICS OF EQ (1.1)

In this section, we investigate the dynamics of (0.1) under the assumptions that all parameters in the equation are positive and the initial conditions are non-negative.

The change of variables $x_n = \left(\frac{\beta}{\gamma}\right)^{1/4} y_n$ reduces (0.1) to the difference equation

$$y_{n+1} = \frac{r y_{n-3}}{1 + y_n y_{n-1} y_{n-2} y_{n-3}} \qquad (0.5)$$

where $r = \frac{\alpha}{\beta}$ and $n = 0,1,\ldots$. It is clear that (0.5) has the equilibrium points $\bar{y} = 0$ and $\bar{y} = (r-1)^{1/4}$.

Lemma 1. The following statements are true.

i. If $r < 1$, then the equilibrium point $\overline{y_1} = 0$ of (0.5) is locally asymptotically stable.

ii. If $r > 1$, then the equilibrium point $\overline{y_1} = 0$ of (0.5) is a saddle point.

iii. When $r > 1$, then the positive equilibrium point $\overline{y_2} = (r-1)^{\frac{1}{4}}$ of (0.5) is unstable.

Proof. The linearized equation of (0.5) about the equilibrium point $\overline{y_1} = 0$ is

$$z_{n+1} = r z_{n-3}$$

so, the characteristic equation of (0.5) about the equilibrium point $\overline{y_1} = 0$ is

$$\lambda^4 - r = 0$$

hence the proof of (i) and (ii) follows from Theorem 1.

For (iii) we assume that $r > 1$, then the linearized equation of (0.5) about the equilibrium point $\overline{y_2} = (r-1)^{\frac{1}{4}}$ has the form

$$z_{n+1} = -\frac{r-1}{r} z_n - \frac{r-1}{r} z_{n-1} - \frac{r-1}{r} z_{n-2} + \frac{1}{r} z_{n-3}$$

where $n = 0,1,\ldots$. So, the characteristic equation of (0.5) about the equilibrium point $\overline{y_2} = (r-1)^{\frac{1}{4}}$ is

$$\lambda^4 + \frac{r-1}{r}\lambda^3 + \frac{r-1}{r}\lambda^2 + \frac{r-1}{r}\lambda - \frac{1}{r} = 0 \qquad (0.6)$$

It is clear that (0.6) has a root in the interval $(-\infty, -1)$, and so $\overline{y_2} = (r-1)^{\frac{1}{4}}$ is an unstable equilibrium point. This completes the proof.

Theorem 2. Assume that $r < 1$, then the equilibrium point $\overline{y_1} = 0$ of (0.5) is globally asymptotically stable.

Proof. We know by Lemma 1 that the equilibrium point $\overline{y_1} = 0$ of (0.5) is locally asymptotically stable. So let $\{y_n\}_{n=-3}^{\infty}$ be a solution of (0.5). It suffices to show that

$$\lim_{n\to\infty} y_n = 0.$$

Since

$$0 \le y_{n+1} = \frac{r y_{n-1}}{1 + y_n y_{n-1} y_{n-2} y_{n-3}} \le r y_{n-1} < y_{n-1}$$

so

$$\lim_{n\to\infty} y_n = 0.$$

This completes the proof.

Theorem 3. (0.5) has 4 period solutions $\{\ldots,\phi_1,\phi_2,\phi_3,\phi_4,\phi_1,\phi_2,\phi_3,\phi_4,\ldots\}$. While $r = 1$, at least one of these solutions are equal to 0 and at least one of them greater than 0.

Proof. Let $\{\ldots,\phi_1,\phi_2,\phi_3,\phi_4,\phi_1,\phi_2,\phi_3,\phi_4,\ldots\}$ which aren't equal with each other, be 4 period solution of (0.5). Then

$$\phi_1 = \frac{r\phi_1}{1+\phi_1\phi_2\phi_3\phi_4}$$
$$\phi_2 = \frac{r\phi_2}{1+\phi_1\phi_2\phi_3\phi_4}$$
$$\phi_3 = \frac{r\phi_3}{1+\phi_1\phi_2\phi_3\phi_4}$$
$$\phi_4 = \frac{r\phi_4}{1+\phi_1\phi_2\phi_3\phi_4}.$$

We have $\phi_1\phi_2\phi_3\phi_4 = r-1$.
If $r \ne 1$ then,

$$\phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$$

or

$$\phi_1 = \phi_2 = \phi_3 = \phi_4 = \overline{y_2}$$

which is impossible, this is a contradiction.
To complete the proof we use $r = 1$ at above equalities

$$\phi_1 = \frac{\phi_1}{1+\phi_1\phi_2\phi_3\phi_4}$$
$$\phi_2 = \frac{\phi_2}{1+\phi_1\phi_2\phi_3\phi_4}$$
$$\phi_3 = \frac{\phi_3}{1+\phi_1\phi_2\phi_3\phi_4}$$
$$\phi_4 = \frac{\phi_4}{1+\phi_1\phi_2\phi_3\phi_4}.$$

So one of the solutions is certainly equal to 0. Let none of them be greater than 0. If they aren't greater than 0, then all of the solutions are equal to 0. This is contradiction with the assumption. So at least one solution certainly greater than 0. This completes the proof.

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