Construction of Asymmetric Fractional Factorial Designs

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Abstract— **In this paper, a method of constructing Asymmetric Fractional Factorial Designs, (AFFD) is presented. This method is based on the extension of a similar concept for symmetric fractional factorial designs (SFFD). A factorial design consisting of n factors is said to be symmetric if, and only if, each factor has the same number of levels, otherwise it is called and asymmetric factorial design. The confounded interactions, and the corresponding confounded degrees of freedom, were determined. The aliases structures and the class of resolution achieved by the constructed designs were determined. Each design obtained and listed achieved a minimum of Resolution III or higher level.**

Index Terms— **Asymmetric, Confounding, Aliases, Resolution , Symmetric.**

I. INTRODUCTION

Many experiments require the study of the joint effect of two or more factors simultaneously. Hence the factorial experiments are the most frequently used and they are most efficient designs for many investigations. By a factorial experiment, we mean that in each complete trial or replication of the experiment, all possible combinations of the levels of factors are investigated. For example, if there are **a** levels of factor A and **b** levels of factor **B**, then each replicate contains all **ab** treatment combinations. When factors are arranged in factorial experiment they are often said to be crosses (Montgomery, 2005).

The effect of a factor is defined to be the change in response produced by a change in the level of a factor. This is frequently called a main effect because it refers to the primary factors of interest in the experiment. In a factorial experiment, the effects of a number of different factors are investigated simultaneously. The treatment consists of all combinations that can be formed from different factors. (Taguchi, 1987). A factor is a discrete variable used to classify experimental units. For example, "Gender" might be a factor with two levels "male" and "female" and "Diet" might be a factor with three levels "low", "medium" and "high" protein. The levels within each factor can be qualitative, such as "Drug A" and "Drug B", or they may be quantitative such as 0, 10, 20 and 30 mg/kg.

A factorial design is one involving two or more factors in a single experiment. Such designs are classified by the number of levels of each factor and the number of factors. So a 2x2 (2^2) factorial will have two factors each at two levels and a 2x2x2x (2^3) factorial will have three factors each at two levels.

Typically, there are many factors such as gender, genotype, diet, housing conditions, experimental protocols, social interactions and age which can influence the outcome of an experiment. These often need to be investigated in order to determine the generality of a response. It may be important to know whether a response is only seen in, say, females but not males. One way to do this would be to do separate experiments in each sex. This "OVAT" or "One Variable at A Time" approach is, however, very wasteful of scientific resources. A much better alternative is to include both sexes or more than one strain etc. in a single "factorial" experiment. Such designs can include several factors without using excessive numbers of experimental subjects.

Factorial designs are efficient and provide extra information (the interactions between the factors), which cannot be obtained when using single factor designs. For example, consider an experiment on cassava yields with 3 factors. These are fertilizers I, II and III with 2 levels, 5% and 10%. The experiment is described as a 2x2x2 (2^3) factorial experiment. Factorial experiment may be 2^k or 3^k i.e. factors at two levels or k factors at three levels. The above illustration of factorial experiment is an agricultural experiment and it involves three factors namely fertilizer I, II and III. Factorial experiment indeed has it origin in agricultural experiment as evident in the series of research works documented by Yates (1937). Furthermore factorial experiments are now widely used in other areas of human endeavor including industries where several factors at 2 or 3 levels may be considered simultaneously. In the circumstance of several factors, fractional factorial may be economical (Finney,1945).

Factorial experiment have several advantages, they are more efficient than one factor at a time experiments. Furthermore a factorial design is necessary when interactions may be present to avoid misleading conclusions. It will also results in a considerable saving of the time and material devoted to the experiment and lastly it allow effects of a factors to be estimated at several level s of the other factors, yielding conclusions that are valid over a range of experimental conditions. We denoted the main effect as A, B and C and interaction effects as AB, AC, BC and ABC for 2^3 factorial experiments.(Hinkelmann and kempthorne,2005).

One of the disadvantages of factorial experiment is that they can get large very quickly with several levels each of several factors. One technique for reducing the size of the factorial to more manageable levels is fractional replication.

Fractional replication is valuable in very large experiments which a single full replication would be too large for the available resources or in which full replication gives more precision for estimating the main effects than is needed (Banerjee,1977). For example in a 2^6 factorial with 64 treatments each main effect is arranged over 32 combinations

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containing each level of the factor. Often a fractions of the replication may be sufficient to obtain the desired precision in such experiments. A method for handling this proposed by Finney in 1945 allows investigators to handle 5 or more factors at a time in an experiment of fractional size so that the investigator can discover quickly which factors are important in a particular study.
In this paper our objectives is to develop some methods for the construction of fractional factorial designs with special attention on cases where the levels of the factors involved are unequal.
Generally a factorial design consists of several factors and each factor having two or more levels. When the levels of the factor are equal we have symmetric factorial experiments. In practice the level of the factors need not be equal.
A fractional factorial design is useful when we cannot afford even one full replicate of a full factorial design, when the number of treatments required exceeds resources when information is required only on main effects and how-order interaction and also when screening studies are needed to check on many factors(Montgomery,1997).

By notation a 2^{h-p} factorial is a factorial experiment consisting of n factors each factors, having 2 levels and where only $½^{p}$ (2^{n}) runs of the full replicate will be run. Similarly we can define 3^{n-p} and others for symmetric factorial settings. The notation in an asymmetric factorial design will be similarly defined. For example, a 2x3x3 factorial experiment consists of three factors, where factor A has 2 levels and factors B and C has 3 levels each. A full replicate would involve 18 runs while a fractional replicate will be a proper fraction of 18 such as 3, 6, or 9. A 1/3 replicate of the 2x3x3 factorial is the set of treatment combination (000, <u>111</u>, 022, 100, 011, 122) where 2 degree of freedom is confounded with blocks.
The treatment combination (III) is defines as the generator. By fractional replications design, information will be lost in the higher-order interaction and in some interactions and effects too.

Asymmetrical factorial designs were first introduced by Yates (1937). Since then a large number of research workers contributed to their construction. However, their use for experimentation specially, in agricultural research has been limited due to the non availability of suitable designs in smaller number of replications or experimental units. It is because most of the efforts have been to find equally replicated designs balance for the confounded interactions and were being addressed to get individual experiment or for getting them series wise like q x 2^{n} or q x 3^{n}. The experiment is therefore resorting to the use of split or strip-plot design or compromising to the limit of the levels of the factors and rising the symmetrical factorial designs, in their place.

One of the technique used was taking the help of the symmetrical factorial design and obtaining the asymmetrical ones as their fractional replications to accommodate one or two factors of asymmetry.(Das,1960). Use of incomplete block design in combination with a symmetrical factorial design to obtain asymmetrical factorial design with one factor of asymmetry is another technique. However all these required a large number of replications, since balancing was sought for the interactions that make affected in the design. Repetition of some levels of the factor of asymmetry, forming their equal sized groups equal to the number of the level of the factors of symmetry and use these groups as the levels instead to get the design in 2 or 3 replications was another attempt. Another techniques proposed was to use some suitably chosen linear function to replace the combinations of 2 or more factors in 2^{n} symmetrical factorial design and thus introduce the factor of asymmetry (Das,1960). Using one to one or one to many, in terms of fractional replication, association scheme for replacement of combinations of factors of 2^{n} or 3^{n} symmetrical factorial design with the levels of the factor of asymmetrical factorial (Barnerjee, 1977, Malhotra, 1989, Handa 1990) was another technique studied for their construction. It is also known that Extended Group Divisible (EGD) design whenever existent, have orthogonal factorial structure with balance. Several methods of construction of EGD design are available in the literature(Parsad et al 2007 and Gupta etal 2011). The aim of this work is to reduce or minimize the amount of experimental error in our design by constructing an efficient fractional factorial design having in mind that main effect were not confounded and also to compare these designs through their aliases structure so as to ascertain their class of resolution.

II. Research Methodology

The factorial designs are constructed based on the type of the design, whether Symmetric or Asymmetric. The symmetric may be very simple to construct than the asymmetric. To construct a factorial design,other ones. In constructing, we have to make sure that we follow the digital notation and the level. For example for 2^{3} designs, we can have the following: 000,001, 010, 011, 100, 101, 110 and 111.

Since the level is 2, 2 should not appear in the design, when it is 2 we take it to the zero e.g. for 2x3x4, the first digit or factor of this design should have 0 or 1 while the second digit or factor should have 0,1 or 2 and the last digit or factor should have 0,1,2 or 3 etc. The construction used to follow zigzag rotation or cyclic rotation
In our construction, we consider the level of the factors, the size of our block, we also look at the different block size we can have. For this purpose we apply the theorem below.

Theorem 1: $L = \sum_{i=1}^{n} \alpha_i X_i = 0$ (modulus p_i)

Where α_i is the exponent or coefficient of the factor, x_i is the level of the factors and p_i $for\ i = 1, 2, . . . , n$ is the level of the i^{th} factor which depends on the size of the block.. We use this example to illustrate this theorem.

Example: For 2x3x3x3 factorial experiment.
This can be divided into 2 groups as : In block of size 18 and 27. Based on this theorem we now have:

- $X_1 + X_2 + X_3 + X_4 = 0$ (mod 3)
  This is for block size of 18, where we divide into 3 groups
- $X_1 + X_2 + X_3 + X_4 = 0$ (mod 2)
  This is for block size of 27, where we divide into 2 groups.

We continue in this manner for other designs. We then apply this algorithm.

III. ALGORITHM FOR CONSTRUCTION OF FRACTIONAL FACTORIAL DESIGN

i Consider the number of factors and the levels.

ii Represent the level of the factors of 2 levels with 0 and 1 and 3 levels with 0, 1, & 2 . and so on.

iii The 1st block which is the principal block should contain the generator and treatment combination when all factors are at zero level (0).

iv To obtain the next trt combination the generator is added to its self and are continue to add the result obtain to the generator and that continue until we need another generator and the process continue until we obtain the size of that block or the whole design

v After the 1st block has been obtained, we then add to the 1st, 2nd or 3rd factor any suitable member as it is required to obtain the rest of the block in our designs.

IV. ALGORITHM FOR DETERMINING THE CONFOUNDED INTERACTION

i. List all the main factor and interaction on the top or the table e.g. A, B, C, AB, AC, BC and ABC etc

ii. List the fractional part: principal block at the left handside of the table e.g. 000 111 022 etc.

iii. Then where you have 000 (i.e. all factors at low level) trt combination then you allocate the factor level minus one e.g. for 2x3x3, you have 2-1 = 1, 3-1=2 and 3-1=2 Then where there is any factors not at low level you allocate minus 1 (-1) e.g. for 111 we have -1 -1and -1 or 222 we have also -1, -1 and -1. Also for 022 we have 2-1=1, -1 and -1 also for 102 we have -1, 3-1=2 and -1 etc. This is for factor A, B and C for the interaction we obtain their product

iv. We then obtain the total and divide the total with the size of the block to know the interaction that was confounded and its degree of freedom

V. ALGORITHM FOR DETERMINING ALIASES STRUCTURE

i. List the main factor and interaction at the top of the table.

ii. List the fractional part, principal block at the left hand side.

iii. For 0000 i.e. all factor at low level the factor A, B, C and D will have negative signs and for 1011, factor A, B, C and D will have positive, negative, positive and positive signs respectively. Any factor that is not at low level will have positive signs e.g. 1122, all factors have positive signs.

iv. Then we observe all the signs for each factors interactions whenever they are equal either negatively or positively then we say they are aliases of each other.

v. Then we decide which resolution it is.

Construction of fractional factorial design

(1) 2✕3✕3✕3 factorial experiment in block of size 18

$x_1 + x_2 + x_3 + x_4 = 0 \pmod 3$

Block I (Principal Block)	Block II	Block III
0 0 0 0	1 1 0 0	0 1 0 0
0 0 1 2	1 1 1 2	0 1 1 2
0 0 2 1	1 1 2 1	0 1 2 1
0 1 0 2	1 1 1 2	0 2 0 2
0 1 1 1	1 1 2 1	0 2 1 1
0 1 2 0	1 2 0 2	0 2 2 0
0 2 0 1	1 0 0 1	0 0 0 1
0 2 1 0	1 0 2 2	0 0 1 0
0 2 2 2	1 0 1 0	0 0 2 2
1 0 0 2	0 0 0 2	1 1 0 2
1 0 1 1	0 0 1 1	1 1 1 1
1 0 2 0	0 0 2 0	1 1 2 0
1 1 0 1	0 1 0 1	1 2 0 1
1 1 1 0	0 1 1 0	1 2 1 0
1 1 2 2	0 1 2 2	1 2 2 2
1 2 0 0	0 2 0 0	1 0 0 0
1 2 1 2	0 2 1 2	1 0 1 2
1 2 2 1	0 2 2 1	1 0 2 1

(2) 4 x 3 x 2 factorial experiment in block of size 12. $X_1 + X_2 + X_3 = 0 \pmod 2$

Block I: 000 011 020 110 121 ⇦ principal block
220 211 321 101 200
301 310

Block II: 001 010 021 111 120
221 210 320 100 201
300 311

Confounding scheme in asymmetric fractional factorial design.

With the help of the algorithm under the methodology, we determine the confounded Interaction and their degree of Freedom as follows:

Let K – size of the block

$$\text{Degree of freedom} = df = \frac{\text{Total}}{K}$$

(1) 2✕3✕3✕3 factorial experiment in block of size 18. Determine the confounded interaction. 1/3 fractional replicate.

A	B	C	D	A.1	B.1	C.1	D.1	AB	AC	AD	BC	BD	CD	ABC	ABD	ACD	BCD	ABCD
0	0	0	0	1	2	2	2	2	2	2	4	4	4	4	4	4	8	8
0	0	1	2	1	2	-1	-1	2	-1	-1	-2	-2	1	-2	-2	1	2	2
0	0	2	1	1	2	-1	-1	2	-1	-1	-2	-2	1	-2	-2	1	2	2
0	1	0	2	1	-1	2	-1	-1	2	-1	-2	1	-2	-2	1	-2	2	2
0	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1
0	1	2	0	1	-1	-1	2	-1	-1	2	1	-2	-2	1	-2	-2	2	2
0	2	0	1	1	-1	2	-1	-1	2	-1	-2	1	-2	-2	1	-2	2	2
0	2	1	0	1	-1	-1	2	-1	-1	2	1	-2	-2	1	-2	-2	2	2
0	2	2	2	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1
1	0	0	2	-1	2	2	-1	-2	-2	1	4	-2	-2	-4	2	2	-4	4
1	0	1	1	-1	2	-1	-1	-2	1	1	-2	-2	1	2	2	-1	2	-2
1	0	2	0	-1	2	-1	2	-2	1	-2	-2	4	-2	2	-4	2	-4	4
1	1	0	1	-1	-1	2	-1	1	-2	1	-2	1	-2	2	-1	2	2	-2
1	1	1	0	-1	-1	-1	2	1	1	-2	1	-2	-2	-1	2	2	2	-2
1	1	2	2	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1
1	2	0	0	-1	-1	2	2	1	-2	-2	-2	-2	4	2	2	-4	-4	4
1	2	1	2	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1
1	2	2	1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1
	Total			0	0	0	0	0	0	0	0	0	0	0	0	0	9	27

This implies that 9/18 df of BCD and 27/18 df of ABCD are confounded with blocks (2df were confounded)

(2). 4 x 3 x 2 factorial experiment in block of size 12. Determining the confounded interaction. ½ fractional replicate.

	A	**B**	**C**	**AB**	**AC**	**BC**	**ABC**
000	3	2	1	6	3	2	6
011	3	-1	-1	-3	-3	1	3
020	3	-1	1	-3	3	-1	-3
110	-1	-1	1	1	-1	-1	1
121	-1	-1	-1	1	1	1	-1
220	-1	-1	1	1	-1	-1	1
211	-1	-1	-1	1	1	1	-1
321	-1	-1	-1	1	1	1	-1
101	-1	2	-1	-2	1	-2	2
200	-1	2	1	-2	-1	2	-2
301	-1	2	-1	-2	1	-2	2
310	-1	-1	1	1	-1	-1	1
Total	**0**	**0**	**0**	**0**	**4**	**0**	**8**

This implies that 4/12 df of AC and 8/12 df of ABC are confounded with blocks. (1 df is confounded).

Aliases structure for asymmetric fractional factorial design.

(1)2×3×3×3 factorial experiment in block of size 18. Determine the aliases structure.

A B C D A.1 B.1 C.1 D.1 AB AC AD BC BD CD ABC ABD ACD BCD ABCD

0 0 0 0	- - - - + + + + + + - - - - +
0 0 1 2	- - + + + - - - - + + + + + -
0 0 2 1	- - + + + - - - - + + + + + -
0 1 0 2	- + - + - + - - + - + + + + -
0 1 1 1	- + + + - - - + + + - - - - +
0 1 2 0	- + + - - - + + - - - - - - +
0 2 0 1	- + - + - + - - + - + + + + -
0 2 1 0	- + + - - - + + - - - - - - +
0 2 2 2	- + + + - - - + + + - - - - +
1 0 0 2	+ - - + - - + + - - + + + + +
1 0 1 1	+ - + + - + + - - + - - - - -
1 0 2 0	+ - + - - + - - + - - - - - -
1 1 0 1	+ + - + + - + - + - - - - - -
1 1 1 0	+ + + - + + - + - - + + + + +
1 1 2 2	+ + + + + + + + + + + + + + +
1 2 0 0	+ + - - + - - - - + - - - - -
1 2 1 2	+ + + + + + + + + + + + + + +
1 2 2 1	+ + + + + + + + + + + + + + +

A = BCD, B = ACD, C = ABD, D = ABC, AB = CD, AC = BD, AD = BC, I = ABCD. This is resolution IV design

(2) 4 x 3 x 2 factorial experiment in block of size 12 Determine the aliases structure

	A	B	C	AB	AC	BC	ABC
000	-	-	-	+	+	+	-
011	-	+	+	-	-	+	-
020	-	+	-	-	+	-	+
101	+	+	-	+	-	-	-
121	+	+	+	+	+	+	+
220	+	+	-	+	-	-	-
211	+	+	+	+	+	+	+
321	+	+	+	+	+	+	+
101	+	-	+	-	+	-	-
200	+	-	-	-	-	+	+
301	+	-	+	-	+	-	-
310	+	+	-	+	-	-	-

Aliases I = -ABC, A, = -BC, B = - AC, C = -AB. This is resolution III

VI. DISCUSSION

The designs constructed were of Resolution III and IV. Most of the confounded interactions were of 2, 3 or 4 ways interaction. The study observed that the size of our block which determine the fraction of our design plays a vital role in this research. The problem of choice of generator does not arise in our construction because we did not use any generator.

VII. CONCLUSION

Based on the above the main effect are not aliases with each other, this implies that information will not be loss on the main effect, information can only be loss on interactions that are confounded. The R-software packages used here can be used on any three or four factors that is of asymmetric factorial design. We can use this software to construct our designs, determine the confounded interaction and their degree of freedom, obtain the aliases structure and the class of resolution of a particular design.

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