Köthe-Toeplitz And Topological

\[ c_0^2(X,\lambda,p), \ c^2(X,\lambda,p) \text{ and } l_\infty^2(X,\lambda,p) \]

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Abstract— This paper is in continuation of [4]. Here we characterize generalized Köthe-Toeplitz duals of the matrix classes \( c_0^2(X,\lambda,p) \), \( c^2(X,\lambda,p) \) and \( l_\infty^2(X,\lambda,p) \) and by application of these duals of the matrix spaces \( c_0^2(X,\lambda,p) \) and \( c^2(X,\lambda,p) \).

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I. INTRODUCTION

Concerning the notations and terminology and results, we follows [1,3]. Let \( (X, \mathcal{S}) \) be a Hausdorff locally convex topological vector space (lc TVS ) over the field of complex numbers \( C \) and \( X^* \) be its topological dual . We denote \( U \) by the fundamental system of balanced, convexandana absorbingneighbourhoodsofzerovecto \( \theta \) to denote \( g_0 \) to denote the gauge (Minkowski functionals) generating the topology \( \mathcal{S} \) of \( X \).

By a generalized matrix , a generalized double sequence we mean a double sequence \( \bar{x} = (x_{mn}) \) with elements from \( X \). Let \( p = (p_{mn}) \) be a double sequence of strictly positive real numbers and \( \lambda = (\lambda_{mn}) \) be a double sequence of non-zero complex numbers.Throughout the paper we shall take \( p = (p_{mn}) \in l_\infty^2 \), space all bounded scalar double sequences , \( H = H(p) = \mathcal{B}(X^*_{mn},P_{mn}) \) and \( M = M(p) = \max(1,H) \). For \( x \in X \), \( S_{mn}(x) \) denotes the double sequence whose all terms are \( x \), (see [4]).

We now consider the dual system \( \langle X, X^* \rangle \) with respect to the canonical bilinear functional \( \langle x, f \rangle \) which is the value of \( f \in X^* \) at \( x \in X \). If \( A \subseteq X \) then polar of \( A \) is denoted to be By space of vector double sequences \( E(X) \) we mean a vector space of double sequences in \( X \) over \( C \) with respect to coordinatewise addition and scalar multiplication . The double summation \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \) denote by \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \) is taken in the sense \( \lim_{n \to \infty} \sum_{2 \leq m+n \leq N} \).

We take \( X^* \) with the strong topology \( \beta(X^*,X) \) generated by the family \( D^\prime = \{ g_{B^0} : B \in B \} \) where \( B \) is the collection of all bounded sets (or \( \sigma(X, X^*) \)-bounded sets) \( B \) of \( X \), \( B^0 \) is the polar of \( B \) with respect to bilinear form \( \langle x, f \rangle = f(x) \) of the pairing \( \langle X, X^* \rangle \) and for \( f \in X^* \).

\[ g_{B^0}(f) = \text{ sup} \{ |\langle x, f \rangle| : x \in B \} \].

A subset \( A \) of linear functional which are defined on lcTVS \( X \) is called equicontinuous if there exists \( U \subseteq U^0 \) such that \( A \subseteq U^0 \). A locally convex topological vector space \( X \) is said to be sequentially barrelled if every sequence \{ \( f_{mn} \) \} \subseteq X^* which converges to \( \theta \) in \( \beta(X^*,X) \) is equicontinuous . For \( U \subseteq U^0 \), the set \( U^0 \) is balanced ,bounded ,convex and \( \beta(X^*,X) \)-complete subset of \( X^* \). Let \( N(U) = \{ x \in X : g_U(x) = 0 \} \). For \( p = (p_{mn}) \) and \( \lambda = (\lambda_{mn}) \) in [4] we have introduced and studied the following classes :

\[ c_0^2(X,\lambda,p) = \{ \bar{x} = (x_{mn}) : x_{mn} \in X, m,n \geq 1 \text{ and } (g_U(\lambda_{mn} x_{mn}))^p_{mn} \to 0 \text{ as } m+n \to \infty \text{ for each } g_U \in D \} \];

\[ c^2(X,\lambda,p) = \{ \bar{x} = (x_{mn}) : x_{mn} \in X, m,n \geq 1 \text{ and } (g_U(\lambda_{mn} x_{mn} - x))^{p_{mn}} \to 0 \text{ as } m+n \to \infty \text{ for each } g_U \in D \} \];
Köthe-Toeplitz And Topological \( c^2_0(X,\lambda,\mu) \), \( c^2(X,\lambda,\mu) \) and \( l^2_\infty(X,\lambda,\mu) \)

(1.3) \( \tilde{x} = (x_{mn}) : x_{mn} \in X, m, n \geq 1 \) and

\[
\sup_{m,n} (g_{U}(x_{mn} \lambda_{mn}))^{p_{mn}} < \infty \quad \text{for each } g_{U} \in D. 
\]

Then the quotient spaces \( X_U = X/ N(U) \) is a normed space with respect to the norm \( \tilde{g} \) where \( \tilde{g}(x) = g_{U}(x) \), \( x(U) \) being the equivalence class in \( X_U \) corresponding to the element \( x \in X \). The subspace \( X^*_U \) is a Banach space with respect to the norm \( g_{U}(f) = \sup \{ |(x,f)| : x \in U \} \). Further we have

THEOREM 1.1: The Banach space \( (X^*_U, g_{U}) \) is the topological dual of \( (X_U, \tilde{g}_U) \) for each \( U \in U \).

We now define the generalized Köthe-Toeplitz duals i.e., generalized \( \alpha-, \beta-, \text{ and } \gamma- \) duals for a class \( E(X) \) of vector double sequences by

\[ (E(X))^\alpha = \{ \tilde{f} = (f_{mn}) : f_{mn} \in X^*, m, n \geq 1 \text{ and } \sum \sum |\langle x_{mn}, f_{mn} \rangle| < \infty \text{ for all } \tilde{x} = (x_{mn}) \in E(X) \}; \]

\[ (E(X))^\beta = \{ \tilde{f} = (f_{mn}) : f_{mn} \in X^*, m, n \geq 1 \text{ and } \sum \sum |\langle x_{mn}, f_{mn} \rangle| \text{ is convergent for all } \tilde{x} = (x_{mn}) \in E(X) \}; \]

\[ (E(X))^\gamma = \{ \tilde{f} = (f_{mn}) : f_{mn} \in X^*, m, n \geq 1 \text{ and } \sup \sum \sum_{2 \leq m + n \leq N} |\langle x_{mn}, f_{mn} \rangle| < \infty \text{ for all } \tilde{x} = (x_{mn}) \in E(X) \}. \]

**DEFINITION 1.2:** Let \( E(X) \) be a space of vector double sequences. \( E(X) \) is said to be normal if for \( \tilde{x} = (x_{mn}) \in E(X) \) and for every scalar double sequence \( \tilde{\alpha} = (\alpha_{mn}) \) with \( |\alpha_{mn}| \leq 1, m, n \geq 1 \), the double sequence \( \tilde{\alpha} \tilde{x} = (\alpha_{mn} x_{mn}) \in E(X) \). (ii) \( E(X) \) is said to be monotone if it contains the canonical pre – images of all its step spaces (cf.[2]).

On the lines of scalar single sequences [2], we can easily prove:

**THEOREM 1.3:** A space \( E(X) \) of vector double sequences is

(i) normal if and only if \( l^2_\infty(E(X)) \subseteq E(X) \); and

(ii) monotone if and only if \( m^2_0(E(X)) \subseteq E(X) \),

where \( m^2_0 \) is the space of scalar double sequences spanned by all double sequences formed by zeros and ones.

Further we easily get:

**THEOREM 1.4:**

(i) \( (E(X))^\alpha \subseteq (E(X))^\beta \subseteq (E(X))^\gamma \).

(ii) \( (E(X))^\alpha = (E(X))^\beta \) if \( E(X) \) is monotone, and

(iii) \( (E(X))^\alpha = (E(X))^\gamma \) if \( E(X) \) is normal.

**II. KöTHE – TOEPLITZ DUALS**

In this section we characterize \( \alpha-, \beta-, \text{ and } \gamma- \) duals \( c^2_0(X,\lambda,\mu) \), \( c^2(X,\lambda,\mu) \) and \( l^2_\infty(X,\lambda,\mu) \) are normal; and

(ii) \( c^2(X,\lambda,\mu) \) is not monotone.

We now define

\[(2.1) \quad M^2_0(X,\lambda,\mu) = \{ \tilde{f} = (f_{mn}) : f_{mn} \in X^* \}, \]

\(m, n \geq 1\) and for each \( B \in B \) there exists an integer \( K > 1 \) such that \( \sum \sum |\lambda|^{-1} g_{E}^{-1}(f_{mn}) K^{-1/p_{mn}} < \infty \}. \)

**THEOREM 2.1:** If \( X \) sequentially barrelled lcTVS then

\[(c^2_0(X,\lambda,\mu))^\alpha = M^2_0(X^*,\lambda,\mu). \]

**COROLLARY 2.3:** If \( X \) is sequentially barrelled lcTVS then

\[(c^2_0(X,\lambda,\mu))^\beta = c^2_0(X^*,\lambda,\mu) \cap S(X^*,\lambda,\mu)^2 \]

\[(c^2_0(X,\lambda,\mu))^\gamma = c^2_0(X^*,\lambda,\mu) \cap S(X^*,\lambda,\mu)^2 \].

**THEOREM 2.4:** Let \( X \) be sequentially barrelled lcTVS. Then

(i) \( (c^2_0(X,\lambda,\mu))^\alpha = M^2_0(X^*,\lambda,\mu) \cap S(X^*,\mu, l^2_\infty) \]

(ii) \( (c^2_0(X,\lambda,\mu))^\beta = M^2_0(X^*,\lambda,\mu) \cap S(X^*,\lambda, (cs))^2 \)

(iii) \( (c^2_0(X,\lambda,\mu))^\gamma = M^2_0(X^*,\lambda,\mu) \cap S(X^*,\lambda, (bs))^2 \).

**COROLLARY 2.5:** If \( \inf \ p_{mn} > 0 \) and \( X \) is sequentially barrelled lcTVS then
\[
\begin{align*}
\left( c^2_0 \left( X, \lambda, p \right) \right)^\alpha &= \left\langle x, f \right\rangle + \sum \sum \left\langle x_{mn}, f_{mn} \right\rangle \\
\text{where } l^2_1 \left( X^*, \lambda \right) &= \{ f = (f_{mn}) : f_{mn} \in X^*, m, n \}, \\
\sum \sum |\lambda_{mn}|^{-1} g_{B}^{\theta} (f_{mn}) < \infty \text{ for each } B \in B.
\end{align*}
\]

For the next theorem we define
\[
(2.2) M^2_{\infty}(X^*, \lambda, p) = \{ f = (f_{mn}) \in X^*, m, n \geq 1 \text{ such that for each } B \in B \text{ and for each } K > 1, \sum \sum |\lambda_{mn}|^{-1} g_{B}^{\theta} f_{mn} K^{-1/p_{mn}} < \infty \}.
\]

**THEOREM 2.6:** If \( X \) is sequentially barreled lc TVS then
\[
(l^2_{\infty}(X, \lambda, p))^{\alpha} = M^2_{\infty}(X^*, \lambda, p).
\]

Moreover from Lemma 2.1 and Theorems 1.4 and 2.6, we easily get:

**COROLLARY 2.7:** If \( X \) is sequentially barreled lcTVS then
\[
(l^2_{\infty}(X, \lambda, p))^{\beta} = (l^2_{\infty}(X, \lambda, p))^{\gamma} = M^2_{\infty}(X^*, \lambda, p).
\]

**III. CONTINUOUS DUAL**

In the following Theorems, continuous duals of \( c^2_0(X, \lambda, p) \) and \( c^2(X, \lambda, p) \) are characterized by applications of the results concerning Köthe – Toeplitz duals obtained in section 2.

**THEOREM 3.1:** If \( X \) is sequentially barreled lcTVS then the topological dual \( (c^2_0(X, \lambda, p))^* \) of \( (c^2_0(X, \lambda, p)) \) is isomorphic to \( M^2_{\infty}(X^*, \lambda, p) \).

**THEOREM 3.2:** If \( \inf P_{mn} > 0 \) and \( X \) is sequentially barreled lcTVS then \( F \in c^2(X, \lambda, p) \), the topological dual of \( (c^2(X, \lambda, p), \sigma g) \), if and only if there exists \( f \in X^* \) and \( \tilde{f} = (f_{mn}) \in l^2_1(X^*, \lambda) \) such that for each \( \tilde{x} = (x_{mn}) \in c^2(X, \lambda, p) \)
\[
F(\tilde{x}) = \left\langle x, f \right\rangle + \sum \sum \left\langle x_{mn}, f_{mn} \right\rangle.
\]

**REFERENCE**