On Geodesics of 3D Surfaces of Rotations in Euclidean and Minkowskian Spaces

Anis I. Saad, Saif Alislam E. Muhammed, Tarig S. Elmabrouk

***Abstract*— In this paper we review geodesics on surfaces of revolution in classical differential geometry, and prove the analogous result in three-dimensional Minowski space; which does have three types of surfaces of rotations [9]. So there are three types of geodesics on those surfaces. Then we introduce some examples explicitly in both cases; the usual one which Euclidean space and the cases of Minkowskian spaces considering a time-like geodesic. And then we visualize some geodesics on those surfaces using Maple. Which showing how the Euclidean and Minkowskian geodesics are differ.**

***Mathematics Subject Classification*: 51B20, 53A05, 53B30, 53C22.**

***Index Terms*—Clairaut's Theorem, Curves and Surfaces Theory in Euclidean and Minkowskian spaces, Minkowski Space, Surfaces of evolution.**

I. Introduction

In Euclidean space, the geodesics on a surface of revolution can be characterized by mean of Clairauts theorem, which essentially says that the geodesics are curves of fixed angular momentum. A similar result holds for three dimensional Minkowski space for time-like geodesics on surfaces of revolution about the time axis. Furthermore, this result holds for generalizations of surfaces of revolution to those surfaces generated by any one-parameter subgroup of the Lorentz group [see [4]].

The change of signature from Euclidean to Minkowskian geometry results in a fascinating interplay between the two forms of geometry: there a formal algebraic similarity in many aspects of the geometry, coupled to important differences between the two, especially in global situations. The lecture notes of Lopez [4], for example, provide a detailed consideration of many of the aspects of three dimensional Minkowski space. The differences arise in various ways, and in Saad and Robert [9] they were concerned with some of the consequences of the fact that vectors in Minkowski space can be classified as time-like, light-like, or space-like by means of the inner product. Also in previous work of Saad and Robert [8] They consider surfaces of revolution in the situation with the closest analogy to the Euclidean situation, namely that of the time-like geodesics on surfaces obtained by rotating a time-like curve about the t-axis in Minkowski space. There are, of course, other types of revolution Minkowski space [3]. This work will take [8] and [9] into account, and then using Pressely book [7] we provide analogue to approach the geodesic on both cases, which finally give the equation of the geodesic on the surface, then we provide some explicit examples cover all of which cases in both spaces. Therefore, we show that the geodesic explicitly on various surfaces of rotations, and then visualize some geodesics explicitly on chosen examples.

In section II and III we provide background material of geodesics of 3D surfaces of rotations in Euclidean and Minkowskian spaces respectively. Further, introduce the Clairaut's theorem of different cases. In section IV we give each chosen example separately with its visualization and geodesic behavior on each.

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II. GEODESICS

Geodesics on surfaces are curves which are the analogues of straight lines in the plane. Lines can be locally thought of either as shortest curves or more generally straightest curves.

Definition 2.1 A curve $\gamma(s)$ on a surface $\mathbb{S}$ is called a geodesic if $\gamma''(s) = 0$ or $\gamma''(s)$ is perpendicular to the tangent plane.

Equivalently, a curve $\gamma(s)$ on a surface $\mathbb{S}$ is geodesic if $\gamma''(s)$ is normal to the surface. More extensive literature properties and notes about the geodesics on a surfaces can be found in [7], [10], [11] and [12] .

A. *Geodesics on Surfaces of Revolution*

Let $\gamma : I \to S$ be a curve given be $\gamma(s) = (x(u(s), v(s)), y(u(s), v(s)), z(u(s)))$ which an arc-length is parameterized geodesic on a surface of revolution. We need the differential equations satisfied by $(u(s), v(s))$. Denote the differentiation with respect to s by an overdot.

Proposition 2.1.1 [7]. On a surface of revolution, every meridian is a geodesic. And a parallel $u = u_0$ is geodesic if and only if $d\rho/du = 0$ when $u = u_0$.

This proposition only deals with these special cases. To understand the rest of geodesics; we need the following theorem; *Clairaut's Theorem*, which is very helpful for studying geodesics on surfaces of revolution.

As much of the following material will be on how this theorem transfers to other situations we give a detailed exposition of the proof.

Let $\mathbb{S}$ be a surface of revolution, obtained by rotating the curve $x = \rho(u)$, `$y = 0$, $z = h(u)$ about the z-axis, where we assume that $\rho > 0$, and $\rho'(u)^2 + h'(u)^2 = 1$. Then $\mathbb{S}$ is parameterized by:

$$\rho(u, v) = (\rho(u)cosv, \rho(u)sinv, h(u)) \quad (1)$$

and has the first fundamental form

$$I = \begin{pmatrix} 1 & 0 \\ 0 & \rho(u)^2 \end{pmatrix} \quad (2)$$

Or

$$ds = \dot{u}^2 + \rho^2(u)\dot{v}^2$$

Now, From the Lagrangian

$$L = \dot{u}^2 + \rho^2\dot{v}^2 \qquad (3)$$

Obtaining the Euler-Lagrangian equation

$$\ddot{u} = \rho\rho'\,\dot{v}^2\,, \qquad \frac{d}{dt}(\rho\dot{v}^2) = 0,$$

so that $\rho\dot{v}^2$ is a constant of the motion.

If γ is a unit-speed geodesic curve on a surface of revolution $\mathbb{S}$, and σ_u, $\rho^{-1}\,\sigma_v$ are unit vectors tangent to the meridians and the parallels, also they are perpendicular since $F = 0$.

Assuming that the curve given by:

$$\gamma(s) = \sigma(u(s), v(s)) \qquad (4)$$

is a unit-speed and we have

$$\dot{\gamma} = \cos(\psi)\,\sigma_u + \rho^{-1}\sin(\psi)\,\sigma_v \qquad (5)$$

Then

$$\sigma_u \times \dot{\gamma} = \rho^{-1}\sin\psi\;\sigma_u \times \sigma_v. \qquad (6)$$

Since $\dot{\gamma} = \dot{u}\,\sigma_u + \dot{v}\,\sigma_v$, then

$$\sigma_u \times [\,\dot{u}\sigma_u + \dot{v}\,\sigma_v] = \dot{v}\sigma_u \times \sigma_v = \rho^{-1}\,\sin\psi\;\sigma_u \times \sigma_v \qquad . \quad (7)$$

Therefore

$$\dot{v} = \rho^{-1}\sin\psi \qquad \text{or} \qquad \rho\dot{v} = \sin\psi, \qquad (8)$$

and then

$$\rho\,\sin\psi = \rho^2\dot{v}. \qquad (9)$$

But the right hand side shows that $\rho\sin\psi$ is constant (say Ω), since $\frac{d}{dt}(\rho^2\,\dot{v}) = 0$.

The converse; if $\rho\sin\psi$ is constant Ω along a unit-speed curve γ in $\mathbb{S}$. Then

$$\dot{v} = \frac{\sin\psi}{\rho} = \frac{\Omega}{\rho^2}. \qquad (10)$$

Also the equation $L = \dot{u}^2 + \rho^2\,\dot{v}^2$ showed above (3).

$$\dot{u}^2 = L - \frac{\Omega^2}{\rho^2}. \qquad (11)$$

Differentiating (11) with respect to t, we got

$$2\dot{u}\ddot{u} = \frac{2\Omega^2}{\rho^3}\dot{\rho} = \frac{2\Omega^2}{\rho^3}\frac{d\rho}{du}\dot{u}, \qquad (12)$$

$$\therefore\; \dot{u}\left(\ddot{u} - \rho\frac{d\rho}{du}\,\dot{v}^2\right) = 0\,. \qquad (13)$$

The term in brackets must be vanish everywhere on γ.

Because, if the term in brackets does not vanish at some point s_0, then there is a number $\epsilon > 0$. .
Such that $\dot{u} = 0$ for $|s - s_0| < \epsilon$, and γ coincides with the parallel $u = u_o$ when $|t - t_0| < \epsilon$. This is Contrary to the assumption. So

$$\ddot{u} = \rho\frac{d\rho}{du}\,\dot{v}^2. \qquad (14)$$

This establishes Clairaut's theorem as follows, and we observe in passing that all meridians are geodesics.

Theorem 2.1.2. [11]. Let γ be a geodesic on a surface of revolution $\mathbb{S}$, let ρ be the distance function of a point of $\mathbb{S}$ from the axis of rotation, and let θ be the angle between γ and the meridians of $\mathbb{S}$. Then $\rho\sin\theta$ is constant along γ. Conversely, if $\rho\sin\theta$ is constant along some curve γ in the surface, and if no part of γ is part of some parallel of $\mathbb{S}$, then γ is a geodesic.

III. TIME-LIKE GEODESICS OF 3D SURFACES OF ROTATIONS IN $M^{2,1}$

In this section we observe three different types of surfaces of rotation in $M^{2,1}$, and then review that the Clairaut's theorem can carry over to those three types; time-like, space-like and null. So we do the same and proof that as much as Clairaut's theorem does.

A. 3D Surfaces of Rotations in $M^{2,1}$

The surfaces of rotation which embedded in $M^{2,1}$ are classified into three different types upon the generator itself by mean of the inner product. i.e. As in Minkowski space three types of inner product (time-like, space-like and light-like) also the surfaces of rotations have three types of generators which
means the axis of rotation. [see [[3],[4],[9]].

Case (1) If the axis of rotation is the time-like axis t and without loss of generality we may assume that the curve γ is lies in the xt- plane. Then the surface of rotation in this case can parameterize by

$$\sigma_t(u, v) = (\cos v\,, \rho(u)\sin v\,, t(u)). \qquad (15)$$

where $\rho(u)$and $t(u)$ are smooth functions.

Case (2) If the axis of rotation is the space-like axis, without loss of generality, we have either the curve γ is located in the xy-plane or yt- plane, then the surfaces of rotation around space-like axis can be parameterized by:

$$\sigma_s(u, v) = (\rho(u)\cosh v\,, q(u), \rho(u)\sinh v), \qquad (16)$$

or

$$\sigma_s(u, v) = (\rho(u)\sinh v\,, q(u), \rho(u)\cosh v)\,. \qquad (17)$$

respectively.

Case (3) Finally, if the axis of rotation is the line spanned by the light-like line of the plane xt, then the surface of rotation can be parameterized by

$$\sigma_n(u, v) = \left(-vy(u) + vt(u), \left(1 - \frac{v^2}{2}\right)y(u) + \frac{v^2}{2}t(u), -\frac{v^2}{2}y(u) + (1 + \frac{v^2}{s}\,t(u)\right)$$

. (18) The parameterizations (15) and (16) give the same surface of rotation but two different parameterizations.

B. Clairaut's Theorem in $M^{2,1}$

In this section, we will revise the Clairaut's theorem to the surfaces of rotations above. And illustrate that in each case we have Clairaut's theorem in Minkowski space.

Here we will generalize the first surface of rotation which is (15).

$$\sigma(u, v) = \left(\rho(u)\cos v\,, \rho(u)\sin v, t(u)\right),\; u \in I, 0 \le v \le 2\pi$$

. (19)

The rotation of the curve is still about Euclidean plane. Also the functions ρ, t have the property of $(\rho')^2 - (t')^2 = -1$ for the time- like condition.

First fundamental form of $\mathbb{S}$ has signature $(-,+)$ everywhere if generating curve is time- like.
Since

$$I_{M^{2,1}} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} g(\sigma_u, \sigma_u) & g(\sigma_u, \sigma_v) \\ g(\sigma_u, \sigma_v) & g(\sigma_v, \sigma_v) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \rho^2 \end{bmatrix}.$$

The geodesics on $\mathbb{S}$ are time-like, and given by:

$$\gamma(s) = \Big(x\big(u(s), v(s)\big), y\big(u(s), v(s)\big), t\big(u(s)\big)\Big), \quad (20)$$

which is an arc-length parameterized geodesic.
If γ is a unit-speed geodesic time-like curve on a surface of revolution $\mathbb{S}$. And $\sigma_u, \rho^{-1}\sigma_v$ are unit vectors tangent to the meridians and the parallels, also they are perpendicular since $F = 0$.

Assuming that the curve γ given by:

$$\gamma(u) = \sigma\big(u(u), v(u)\big), \quad (21)$$

is a unit-speed, and we have

$$\dot{\gamma} = \cosh(\psi)\, \sigma_u + \rho^{-1}\sinh(\psi)\sigma_v. \quad (22)$$

Then

$$\sigma_u \times \dot{\gamma} = \rho^{-1} \sinh\psi\ \sigma_u \times \sigma_v, \quad (23)$$

where $\times$ is the 3D Minkowski cross product.
Since $\dot{\gamma} = \dot{u}\sigma_u + \dot{v}\sigma_v$, then

$$\sigma_u \times [\dot{u}\sigma_u + \dot{v}\sigma_v] = \dot{v}\sigma_u \times \sigma_v = \rho^{-1} \sinh\psi\ \sigma_u \times \sigma_v. \quad (24)$$

Therefore

$$\dot{v} = \rho^{-1} \sinh\psi \quad \text{or} \quad \rho\dot{v} = \sinh\psi \ , \quad (25)$$

and then

$$\rho \sinh\psi = \rho^2 \dot{v}. \quad (26)$$

But the right hand side shows that $\rho \ \sinh\psi$ is constant (say Ω), $\frac{d}{dt}(\rho\, \dot{v}^2) = 0$.

The converse if $\rho \sinh\psi$ is constant Ω along a unit-speed curve γ in $\mathbb{S}$. Then

$$\dot{v} = \frac{\sinh\psi}{\rho} = \frac{\Omega}{\rho^2}. \quad (27)$$

Also, from Lagrangian equation, we have $L = \|\dot{\gamma}\|^2 = -\dot{u}^2 + \rho^2\, \dot{v}^2$. So

$$\dot{u}^2 = \frac{\Omega^2}{\rho^2} - L. \quad (28)$$

Differentiating (28) with respect to s, we got

$$2\dot{u}\ddot{u} = \frac{-2\Omega^2}{\rho^3}\dot{\rho} = \frac{-2\Omega^2}{\rho^3}\frac{d\rho}{du}\dot{u}\ , \quad (29)$$

$$\therefore \dot{u}\left(\ddot{u} + \rho\frac{d\rho}{du}\dot{v}^2\right) = 0. \quad (30)$$

The term in brackets must be vanish everywhere on γ.

Because, if the term in brackets does not vanish at some point s_0, then there is a number $\epsilon > 0$.
Such that $\dot{u} = 0$ for $|t - t_0| < \epsilon$, and γ coincides with the parallel $u = u_0$ when $|s - s_0| < \epsilon$. This is contrary to the assumption. So

$$\ddot{u} = -\rho\frac{d\rho}{du}\dot{v}^2 \quad (31)$$

So, if we do the same procedure and conclusion we have for each case, the Clairaut's theorem can carry over to all cases of 3D Minkowski space [9]. Here we are interesting in providing some explicit examples satisfy the theorem.

IV. EXAMPLES AND VISUALIZATIONS

In this part of the paper, we present some examples of surfaces of revolution covering E^3 and $M^{2,1}$. And visualize the geodesics for a given surfaces in all cases.

A. Example of visualization of surface in E^3

We use the Clairaut's theorem to determine the geodesics on surface of revolution, say pseudo-sphere in E^3(see [7], p 230).

$$\sigma(u, v) = \left(\exp(u)\cos v, \exp(u)\sin v, \sqrt{1 - \exp(2u)} - \cosh^{-1}(\exp(-u))\right) \quad (32)$$

The first fundamental form is given by

$$du^2 + \exp(2u)\, dv^2. \quad (33)$$

Let us use $w = \exp(-u)$, and re-parameterized the surface. It would be convenient. So the new parameterization of the surface is:

$$\tilde{\sigma}(w, v) = \left(\frac{1}{w}\cos v\, ,\ \frac{1}{w}\sin v\, ,\ \sqrt{1 - \frac{1}{w^2}} - \cosh^{-1} w\right). \quad (34)$$

Also the first fundamental form is :

$$\frac{dv^2 + dw^2}{w^2}. \quad (35)$$

We have $w > 1$ for $\tilde{\sigma}$ to be well defined and smooth. If we have a unit-speed geodesic $\gamma(t) = \tilde{\sigma}(v(t), w(t))$, then the unit condition gives:

$$\dot{v}^2 + \dot{w}^2 = w^2. \quad (36)$$

Here, Clairuat's theorem gives:

$$\frac{1}{w}\sin\psi = \frac{1}{w^2}\dot{v} = \Omega, \quad (37)$$

Where Ω is a constant, since $\rho = {}^{1}/_{w}$. Therefore, $\dot{v} = \Omega w^2$. If $\Omega = 0$, we get v is constant. And if $\Omega \neq 0$ and the above equation gives

$$\dot{w} = \pm w\sqrt{1 - \Omega^2 w^2}. \quad (38)$$

Hence, along the geodesic,

$$\frac{dv}{dw} = \frac{\dot{v}}{\dot{w}} = \pm\frac{\Omega w^2}{w\sqrt{1-\Omega^2 w^2}} = \pm\frac{\Omega w}{\sqrt{1-\Omega^2 w^2}}. \quad (39)$$

So

$$(v - v_0) = \mp\frac{1}{\Omega}\sqrt{\Omega^2 w^2}. \quad (40)$$

Then

$$(v - v_0)^2 + w^2 = \frac{1}{\Omega^2}\ , \quad (41)$$

where v_0 is constant.

So the geodesic is parts from circles in the vw-plane, and lying in the region $w > 1$. Moreover all these circles have centre on the v-axis, and insect in the v-axis perpendicularly. The meridians correspond to straight lines perpendicular to the v-axis.

Now we need to visualize these parts of circles on the Pseudo-sphere to see the geodesics on the pseudo-sphere. From our supposition, we have $w = \exp(-u)$, then $u = -\ln w$. Also,

$$w = \frac{1}{\Omega}\sin s \tag{42}$$

$$v = \frac{1}{\Omega}\cos s + v_0 \tag{43}$$

$$\left.\begin{aligned} u &= -\ln w = -\ln\left(\frac{1}{\Omega}\sin s\right) \\ &= \ln\Omega - \ln(\sin s) = \ln\left(\frac{\Omega}{\sin s}\right) \end{aligned}\right\} \tag{44}$$

$$v = \frac{1}{\Omega}\cos s + v_0. \tag{45}$$

So the curve on the pseudo-sphere is:

$$X_{E^3}(s) = \left(\rho\left(\ln\left(\frac{\Omega}{\sin s}\right)\sin\left(\frac{1}{\Omega}\cos s + v_0\right)\right), \rho\left(\ln\left(\frac{\Omega}{\sin s}\right)\cos\left(\frac{1}{\Omega}\cos s + v_0\right)\right), z\left(\ln\left(\frac{\Omega}{\sin s}\right)\right)\right). \tag{46}$$

Then the geodesic on the pseudo-sphere will be

$$X_{E^3}(s) = \left(\frac{\Omega}{\sin s}\sin\left(\frac{1}{\Omega}\cos s + v_0\right), \frac{\Omega}{\sin s}\cos\left(\frac{1}{\Omega}\cos s + v_0\right), \sqrt{1-\left(\frac{\Omega}{\sin s}\right)^2} - \cosh^{-1}\left(\frac{\sin s}{\Omega}\right)\right). \tag{47}$$

Now, we are ready to visualize the geodesic on the pseudo-sphere.

The corresponding geodesics on the pseudo-sphere itself are shown below (figure(1)).

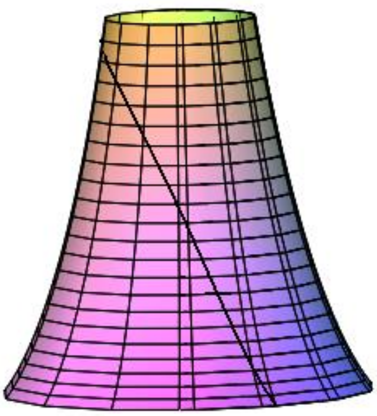


Figure (1)

B. Example of Visualization of Surface in $M^{2,1}$ with Time-like Rotation

Here we use the Clairaut's theorem to determine the geodesics on pseudo-sphere in $M^{2,1}$, such that the rotation line is time-like.

$$\sigma(u, v) = \left(\exp(u)\cos v, \exp(u)\sin v, \sqrt{1-\exp(2u)} - \cosh^{-1}(\exp(-u))\right). \tag{48}$$

The first fundamental form is given by

$$du^2 - \exp(2u)\, dv^2. \tag{49}$$

Lets again use $w = \exp(-u)$, and reparametrized the surface. So the new parametrization of the surface is:

$$\tilde{\sigma}(u, v) = \left(\frac{1}{w}\cos v, \frac{1}{w}\sin v, \sqrt{1-\frac{1}{w^2}} - \cosh^{-1} w\right) \tag{50}$$

Also the first fundamental form is :

$$\frac{dv^2 - dw^2}{w^2}. \tag{51}$$

We have $w > 1$ for $\tilde{\sigma}$ to be well defined and smooth. If we have a unit-speed geodesic $\gamma(t) = \tilde{\sigma}(v(t), w(t))$, then the unit condition gives:

$$\dot{v}^2 - \dot{w}^2 = -w^2. \tag{52}$$

Here, Clairuat's theorem gives:

$$\frac{1}{w}\sinh\psi = \frac{1}{w^2}\dot{v} = \Omega, \tag{53}$$

Where Ω is a constant, since $\rho = 1/w$. Therefore, $\dot{v} = \Omega w^2$. If $\Omega = 0$, we get $v =$ constant. And if $\Omega \neq 0$ and the above equation gives

$$\dot{w} = \pm w\sqrt{\Omega^2 w^2 - 1}. \tag{54}$$

Hence, along the geodesic,

$$\frac{dv}{dw} = \frac{\dot{v}}{\dot{w}} = \pm\frac{\Omega w}{\sqrt{\Omega^2 w^2 - 1}}, \tag{55}$$

so

$$(v - v_0) = \mp\frac{1}{\Omega}\sqrt{\Omega^2 w^2 - 1} \tag{56}$$

then

$$w^2 - (v - v_0)^2 = \frac{1}{\Omega^2}, \tag{57}$$

where v_2 is constant.

So the geodesic is parts from hyperbolas in the vw-plane, and lying in the region $w > 1$.
Moreover all these hyperbolas insect in the v-axis .

Now we need to visualize these parts of hyperbolas on the Pseudo-sphere to see the geodesics on the pseudo-sphere. Again, from our supposition, we have $w = \exp(-u)$, then $u = -\ln w$. Also,

$$w = \frac{1}{\Omega}\cosh s \tag{58}$$

$$v = \frac{1}{\Omega}\sinh s + v_0. \tag{59}$$

Then, the parameters curve in (u, v) space,

$$\left.\begin{aligned} u &= -\ln w = -\ln\left(\frac{1}{\Omega}\sin s\right) \\ &= \ln\Omega - \ln(\cosh s) = \ln\left(\frac{\Omega}{\cosh s}\right) \end{aligned}\right\} \tag{60}$$

$$v = \frac{1}{\Omega}\sinh s + v_0. \tag{61}$$

So the curve on the pseudo-sphere is:

$$\tilde{X}_{M^{2,1}}(s) = \left(\rho\left(\ln\left(\frac{\Omega}{\cosh s}\right)\sin\frac{1}{\Omega}\sinh s + v_0\right), \rho\left(\ln\left(\frac{\Omega}{\cosh s}\right)\cos\left(\frac{1}{\Omega}\sinh s + v_0\right)\right), t\left(\ln\left(\frac{\Omega}{\cosh s}\right)\right)\right) \quad (62)$$

Then the geodesic on the pseudo-sphere will be:

$$\tilde{X}_{M^{2,1}}(s) = \left(\frac{\Omega}{\cosh s}\sin\left(\frac{1}{\Omega}\sinh s + v_0\right), \frac{\Omega}{\cosh s}\cos\left(\frac{1}{\Omega}\sinh s + v_0\right), \sqrt{1-\left(\frac{\Omega}{\cosh s}\right)^2} - \cosh^{-1}\left(\frac{\cosh s}{\Omega}\right)\right). \quad (63)$$

Now, we are ready to visualize the geodesic on the pseudo-sphere.

The corresponding geodesics on the pseudo-sphere itself are shown below (figure (2)).

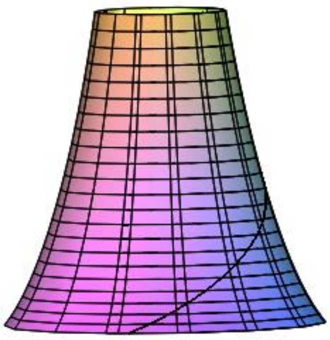


Figure (2)

C. *Example of Visualization of Surface in $M^{2,1}$ with Space-like Rotation*

Now, if we are doing the same with other cases, for example the surface of rotation which generated by space-like rotation, we have chosen an example to determine the geodesics on the surface in $M^{2,1}$, such that the rotation line is y- axis. We have chosen a surface of revolution very carefully to get solvable ordinary differential equation. Therefore the surface is:

$$\sigma(u,v) = \left(\exp(u)\cosh v, \frac{1}{2}\exp(u)\sqrt{1-\exp(2u)} + \frac{1}{2}\tan^{-1}\left(\frac{\exp(-u)}{\sqrt{1-\exp(2u)}}\right), \exp(u)\sinh v\right), \quad (64)$$

which has first fundamental form is given by

$$du^2 - \exp(2u)dv^2. \quad (65)$$

With the same calculation, we have the geodesic equation given:

$$\tilde{X}_{M^{2,1}}(s) = \left(\frac{\Omega}{\sinh s}\cosh\left(\frac{1}{\Omega}\cosh s + v_0\right), \frac{1}{2}\frac{\Omega}{\sinh s}\sqrt{1-\left(\frac{\Omega}{\sinh s}\right)^2} + \frac{1}{2}\tan^{-1}\left(\frac{\frac{\Omega}{\sinh s}}{\sqrt{1-\left(\frac{\Omega}{\sinh s}\right)^2}}\right), \frac{\Omega}{\sinh s}\sinh\left(\frac{1}{\Omega}\cosh s + v_0\right)\right) \quad (66)$$

The corresponding geodesics on this surface itself are shown below (figure(3)).

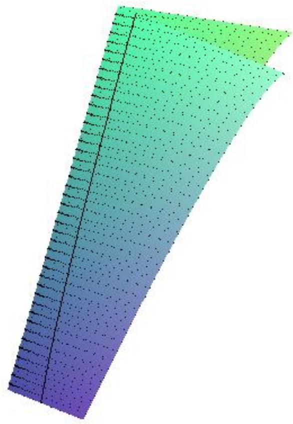


Figure (3)

D. *Examples of Surface in $M^{2,1}$ with Light-like (Null) Rotation*

We will give an example to determine the geodesics on surface of revolution in $M^{2,1}$, generated by null rotation. We have chosen a surfaces of revolution which is null rotation and satisfied the conditions above of the first fundamental form. So the surface could parameterize by:

$$\sigma(u,v) = \left[-v\cosh u + v\sinh u, \left(1-\frac{v^2}{2}\right)\cosh u + \frac{v^2}{2}\sinh u, \frac{-v^2}{2}\cosh u + \left(1+\frac{v^2}{2}\right)\sinh u\right] \quad (67)$$

That has the first fundamental form of

$$-du^2 + \rho^2(u)dv^2 \quad (68)$$

Such that, $\rho(u) = y(u) - t(u) = \cosh u - \sinh u$.

From Clairaut's theorem, we got

$$\dot{v} = \frac{\Omega}{\rho^2(u)}, \quad (69)$$

so,

$$\dot{u} = \pm\sqrt{1+\frac{\Omega^2}{\rho^2(u)}}. \quad (70)$$

Therefore, we have the following ODE

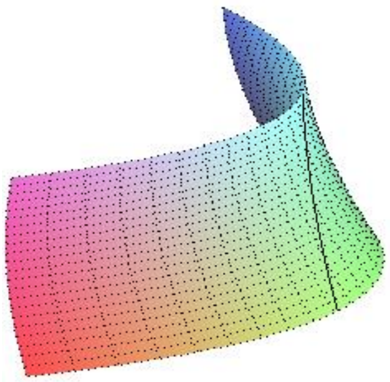
$$\frac{\dot{v}}{\dot{u}} = \frac{dv}{du} = \frac{\frac{\Omega}{\rho^2(u)}}{\sqrt{1+\frac{\Omega^2}{\rho^2(u)}}}, \quad (71)$$

with solving the above equation numerically using $\Omega = 1$, with Maple we get

$$(v - v_0) = \pm\left(-\sqrt{2} + \sqrt{\exp(2u)+1}\right), \quad (72)$$

which the geodesic equation and v_0 is constant.

Plotting the surface chosen and the geodesic equation which solved numerically. And take the positive part of real line in u-axis. We have (see figure (4))



V. CONCLUSION AND FUTURE WORK

To sum up, the geodesics vary from Euclidean to Minkowskian spaces; we can demonstrate that all geodesics on surfaces of rotation in Euclidean case are given by circles; while all geodesics of all different cases of surfaces of rotation in Minkowskian space are given by hyperbolas, whatever the parameterization.

In the future work, we will try to generalize Clairaut's theorem including the possibility of the geodesics being space-like or null. Moreover, the Jacobi Field: is a vector field along a geodesic describing the difference between the geodesic and an "infinitesimally close" geodesic. We will try to think about geodesic deviation in surfaces of rotation in Minkowski space.

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