On Calculus of Manifolds with Special Emphasis of 3D Minkowski Space $M^{2,1}$

Anis I Saad

Abstract— **In this paper, we explain some topics of calculus of manifold, especially for the spacetime symmetry topic. With emphasis of 3D Minkowski differential geometry. The most important symmetries are $g_{ij} = g_{ij}$, A diffeomorphism of this symmetry is called the isometry. If a one-parameter group of isometries is generated by a vector field V, then "this vector field is called a Killing vector field. Which shows that the Lei derivative is vanishing" [14]. Moreover the one parameter group of diffeomorphism called the flow. However the Poincare' group "is the group of isometries of Minkowski spacetime. Also "it is a full symmetry of special relativity includes the translations , rotation and boosts" [11].**

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I. INTRODUCTION

Manifolds are topological spaces that locally have the structure of a coordinate space R^n. They are found in almost all parts of modern mathematics. The aim of this publication is studying the calculus of manifold with emphasis of $M^{2,1}$. And will take into account to cover the isometry ,which is a distance-preserving map between metric spaces. Also the topic of symmetry in spacetime will be provided.

In section 2 we will start with the pull back and push forward functions. And section 3 will take the isometry definition. Section 4 will talk about the Lie Algebra and Lie brackets. After that section 5 will study the functions of flows and integral curves. And section 6 will discuss Poincare' groups. And finally sections 7 and 8 will be finished with the Lie derivative and Killing vector fields with the relationship between them.

II. PULL BACK AND PUSH FORWARD

Definition Let $\rho: N \to M$ be a smooth map between two manifolds. May
not have the same dimension. And let f be a smooth function on M. Its *pull-back* under the map f is the function $\rho^* f = (f \circ \rho)$.

Definition Let N and M be two manifolds, and let $: N \to M$, $P \longmapsto Q$ the *push-forward* of a vector $v \in T_p N$ is a vector $f_* v \in T_Q M$ defined by:

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$$f_* v(g) = v(g \circ f)$$

for all smooth functions $g: M \to \mathbb{R}$, therefore we can write

$$f_* v(g) = v(f^* g).$$

The pushforward has the linearity property:

$$f_*(v_1 + v_2) = f_* v_1 + f_* v_2$$
$$f_*(\lambda v) = \lambda f_* v.$$

And if M_1,M_2,M_3 are three manifolds with maps $f: M_1 \to M_2$, $f: M_2 \to M_3$, it follows that:

$$(g \circ f)_* = g_* f_*,$$

i.e.

$$(g \circ f)_* = g_* f_* v \quad \forall v \in T_p M_1.$$

III. THE ISOMETRY

The *isometry* is a function that preserves a metric, either in metric space or in the topic of Riemannian maniod. In manifolds the isometry f between two maniods (say Reimannian manifolds) is a morphism function.

The most important symmetries of the metric, for which

$$\emptyset_* g_{\mu v} = g_{\mu v}.$$

A diffeomorephism of this type is called an isometry.

Definition A diffeomorphisom $\emptyset: (M, g) \to (N, h)$ is an isometry if $\emptyset^* h = g$.

Last definition means that, for any diffeomorphism $\emptyset$ and for evey point $x, d_x \emptyset$. is linear isometry between $T_x M$ and $T_{\emptyset(x)} N$.

The *isometry* group of (M, g) "is the set of diffeomorphism of M that are g- isometric" [16]. The Lorentz group is the group of isometry of M which fix the origin.
A local diffeomorphism $\emptyset$ with $\emptyset^* h = g$ is a local isometry.

Another definition for the isometry that, if $f: M \to M$, is a diffeomorephism
from maniod to itself, with the property that, $\forall\, p \in M$ and all $V, W \in T_p M$

$$g(Df_p V, Df_p W) = g(V, W),$$

then f is said to be an isometry of (M, g) .

In terms of local coordinates, then

$$g_{ab}(f(p)) \frac{\partial f^a}{\partial x^c} V^c \frac{\partial f^b}{\partial x^d} W^d = g_{cd}(p) V^c W^d,$$

and since this holds for all V, W, then the condition for an isometry in local coordinates is

$$g_{ab}(f(p)) \frac{\partial f^a}{\partial x^c} \frac{\partial f^b}{\partial x^d} = g_{cd}(p).$$

A generic Riemannian manifold has no isometries other than the identity
map.
The availability of the isometry is equivalent to the

availability of the symmetry.
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Example: In 3D Minkowski Space, $(M,g) = (\mathcal{M}^{2,1}, g_{ij})$,where g_{ij} is the 3D Minkowski metric given by:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

There are many parameter of isometry groups- the Poincare' group for example; see section 6.

IV. Lie Brackets and Lei Algebra

A. Lie Brackets

Lie Brackets plays an important role in differential geometry and differential topology. Also "it is a fundamental in the geometric theory for nonlinear control system" [4].

In mathematical field of differential geometry. The Lie bracket of vectors or the commutator is a bilinear differential operator which assigns, to any two vector fields u and v on a smooth manifold M, a third vector field denoted $[u,v]$.

Let a smooth function $f, v(f)$ is smooth function on M. And let two vector fields u, v. Then $u(v(f))$ is also a smooth function , linear in f.

Now, consider

$$\begin{aligned} u\big(v(fg)\big) &= u(fv(g) + v(f)g) \\ &= u(f)v(g) + fu\big(v(g)\big) + u\big(v(f)\big)g + v(f)u(g). \end{aligned}$$

Now, reorder the terms to get:

$$uv(fg) = fuv(g) + uv(f)g + u(f)v(g) + v(f)u(g),$$

So, Leibniz rule is not satisfied by vu, we get

$$vu(fg) = fvu(g) + vu(f)g + v(f)u(g) + u(f)v(g),$$

So, if we subtracting uv and vu, we have

$$(uv - vu)(fg) = f(uv - vu)(g) + (uv - vu)(f)g,$$

This combination means:
$[u,v] \coloneqq uv - vu$
is also a vector field on M. Also this combinations is so-called the **commutator** or **Lie bracket** of the vector field u and v.

Properties of the Lie brackets:
1. The Lie brackets is antisymmetric, $[u,v] = -[v,u]$
2. Also satisfies the **Jacobi identity**
$\big[[u,v],w\big] + \big[[v,w],u\big] + \big[[w,u],v\big] = 0.$

The Lie bracket is useful for computation of Lie derivative and Killing vector field.

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Example: A simple example will be introducing here, is the polar coordinate system of $\mathbb{R}^2$. The unit vectors are

$$e_r = e_x \cos\theta + e_y \sin\theta$$
$$e_\theta = -e_x \cos\theta + e_y \sin\theta,$$

with $e_x = \frac{\partial}{\partial x}$ and $e_y = \frac{\partial}{\partial y}$ being the Cartesian coordinate basis vector, and

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}, \quad r = \sqrt{x^2 + y^2}$$

We know $[e_x, e_y] = 0$, and we want to compute $[e_r, e_\theta] = 0$ and that by take $e_r e_\theta(f(x,y))$ and $e_\theta e_r(f(x,y))$ it is easy to see that $e_r e_\theta\big(f(x,y)\big) \neq e_\theta e_r(f(x,y))$.

And that confirms that, in this case$[e_r, e_\theta] \neq 0$.
Therefore $\{e_r, e_\theta\}$ do not form a coordinate basis.

B. Lie Algebra

A *Lie Algebra* L, is a vector space over some field together with a bilinear multiplication of $L \times L \to L$, is a Lie bracket on L, which satisfies two simple properties:
1- Antisymmetric $[x,y] = -[y,x]$
2- Jacobi identity

$$\big[[x,y],z\big] + \big[[y,z],x\big] + \big[[z,x],y\big] = 0.$$

The Jacobi identity is not really identity - it does not hold for an arbitrary algebra - But it must be satisfied by an algebra for it to be called a Lie algebra.

Example A simple example is the vector space $\mathcal{M}^{2,1}$ equipped with the cross-product

$$v \wedge w = (v_3 w_2 - v_2 w_3, v_1 w_3 - v_3 w_1, v_1 w_2 - v_2 w_1).$$

V. Integral Curves and Local Flows

A. Integral curves

In this section, we start work into vector field deeply. The first objects with vector fields are the integral curves, "which are soomth curves whose tangent vector at each point is the value of the vector field there" [8].

Definition An **integral curve** of a vector field V is a curve γ in a manifold M such that its tangent vector at each point is V .

Or If V is a smooth vector field on M, an **integral curve** of V through the point $p \in M$is a smooth curve $\gamma: I \to M$ such that
$\gamma(0) = p$ and
$\gamma'(t) = V_{\gamma(t)} \quad \forall\, t \in I \ \in \mathbb{R}.$

Example Here we illustrate a simple example in $\mathbb{R}^3$, that if $M = \mathbb{R}^3$ with coordinates (x,y,z) and the vector field given by $V = \frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y}$

The derivative of γ of the curve γ given by:

$$\gamma'(t) = \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z}$$

So the equation for an integral curve of V is:

$$\frac{\partial x}{\partial t} = 1$$

$$\frac{\partial y}{\partial t} = 2t$$

$$\frac{\partial z}{\partial t} = 0$$

This gives our curve

$$\gamma(t) = (t + a_1, t^2 + a_2, a_3).$$

Where a_1, a_2 and a_3 are arbitrary constant.

VI. LOCAL FLOWS

"The collection of all integral curves of a given vector field on a manifold determines a family of diffeomorphisms of (open subsets of) the manifold, called a **flow**." [8].

For any neighbourhood U of p in a manifold M, we have γ_Q, the integral curve through Q. So we can define a map $\emptyset: I \times U \to M$ given by $\emptyset(t,Q) = \gamma_Q(t)$ Where $\gamma_Q(t)$ satisfies

$$\frac{d}{dt} x^i\left(\gamma_Q(t)\right) = v\left(x^i\left(\gamma_Q(t)\right)\right),$$
$$\gamma_Q(0) = Q.$$

Definition The **Local flow** of v is defined by the map $\emptyset_t: U \to M$ at each t given by $\emptyset_t(Q) = \emptyset(t,Q) = \gamma_Q(t)$.

Or the map $\emptyset_t$ for any t is taking any point by parameter distance t along the curve $\gamma_Q(t)$.

The local flow has the following properties:

1. $\emptyset_0$ is the identity map of U
2. $\emptyset_{t+s} = \emptyset_t \circ \emptyset_s$ for all $s, t, s+t \in U$
3. Each flow is a diffeomorphism with $\emptyset_t^{-1} = \emptyset_{-1}$.

The second property follows from the uniqueness from integral curve.

The properties above with the local flow define the **One-parameter groups of diffeomorphisms.**

Example A simple example will be provided here, just for more explanation, That if $\emptyset_t(x) = x + t$, than easy to check the properties above holds.

$$\emptyset_{t+s}(x) = x + t + s = x + s + t = (x + s) + t$$
$$= \emptyset_t(x + s) = \emptyset_t(\emptyset_s(x))$$
$$= \emptyset_t \circ \emptyset_s(x).$$

VII. POINCARE' GROUPS

" The Poincare' group, named after Henri Poincare', is the group of isometries of Minkowski spacetime." [11].

Definition The **Poincare' group** is a semidirect product of the translations and the Lorentz transformation :

$$\mathbb{R}^{1,3} \rtimes O(1,3).$$

Or, the Poincare' group is a group extension of the Lorentz group by a vector representation of it. Also is defined by the group of extension of the Lorentz group by a vector representation of it. Sometimes called the **inhomogeneous Lorentz Group.**

However, the geometry of Minkowski space could be defined by this group (Poincare' group). Also it is helpfull in calculus of manifolds.

Definition The **Poincare' Algebra** is the Lie algebra of the poincare' group. In component from the poincare' algebra is given by the commutation relation :

- $[P_\mu, P_v] = 0$
- $[M_{\mu v}, P_\rho] = g_{\mu\rho}P_v - g_{v\rho}P_\mu$
- $[M_{\mu v}, M_{\rho\sigma}] = g_{\mu\rho}M_{v\sigma} - g_{\mu\sigma}M_{v\sigma} - g_{v\rho}M_{\mu\sigma} + g_{v\sigma}M_{\mu\rho}$

Where P is the generator of translation, M is the generator of Lorentz transformation, which given for *rotation* by
$M_{yx} = x\partial y - y\partial x$
and for *boost* by
$M_{xt} = t\partial x + x\partial t$

and g is the Minkowski metric, (given above in section 3).

Example In $\mathcal{M}^{2,1}$ for (x, y, t) then,

- $[P_x, P_y] = \frac{\partial}{\partial x}\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\frac{\partial}{\partial x} = 0.$
- $[M_{yt}, P_y] = (t\partial y - y\partial t)\partial y - \partial y(t\partial y - y\partial t) = P_t.$
- $[M_{yt}, P_x] = (t\partial y - y\partial t)\partial x - \partial x(t\partial y - y\partial t) = 0$
- If B_x and B_y are boosts in direction of x and y respectively. i.e. $B_x = t\partial x + x\partial t$ and $B_y = t\partial y + y\partial t$ then,
  $[B_x, B_y] = (t\partial x + x\partial t)(t\partial y + y\partial t) - (t\partial y + y\partial t)(t\partial x + x\partial t)$

  $= x\partial y - y\partial x$

Which is the rotation of x and y.

In conclusion , we can conclude that the Poincare' group ,"is full symmetry group of any relativistic field theory. As a result, all elementary fall in representations of this group." [11] and it is the full symmetry of special relativity and includes translations , rotations and boosts.

VIII. LIE DERIVATIVE

Let v be a vector field on a smooth manifold M and if $\emptyset_v$ is the local flow generates by . For each $t \in \mathbb{R}$, the map $\emptyset_v$ is diffeomorephism of M and so it induces a push-forward and pull backs,
$(\emptyset_* v)f = (\emptyset^{-1})^* v(\emptyset^*(f))$.
We define the **Lie derivative** of the function f with respect to v by

$$L_v f = \lim_{t\to 0}\left(\frac{\emptyset_t^* f - f}{t}\right) = \frac{d}{dt}\emptyset_t^* f\bigg|_{t=0}$$

Now, since $\emptyset_t^* = f \circ \emptyset_t$, we have that

$$\frac{d}{dt}\emptyset_t^* f\bigg|_{t=0}(p) = \frac{d}{dt}f(\emptyset_t(p))|_{t=0} = \frac{d}{dt}f\left(\gamma_p(t)\right)\bigg|_{t=0}$$
$$= v(p).f \quad . \quad \forall p \in M$$

where the tangent vector to γ_p at p is $X(p)$. We get

$$L_v f = vf.$$

However, if X, Y be two vector fields on . We can define the *Lie derivatives* of Y with respect to X by

$$L_X Y = \lim_{t\to 0}\left(\frac{\emptyset_{-t*}Y - Y}{t}\right) = \frac{d}{dt}\emptyset_{-t*}Y\bigg|_{t=0}$$

where $\emptyset_t$ is generated by X.

Properties of Lie derivative:

1. Lie derivative is a *Linear operator*: i.e. $L_v(f+g) = L_v f + L_v g$, $L_v(\lambda f) = \lambda v L_v f$.
2. Lie derivative satisfies the *Lebnitz identity* i.e. $L_v(fg) = L_v f . g + f L_v g$.
3. The Lie derivative is linearly depends on v : $\forall f \in C^\infty(M), v, w \in X(M), L_{fv} = f L_v , L_{v+w} = L_v + L_w$.

Lemma [13] The Lie derivative of a vector Y with respect to X is just the Lie brackets of X and Y.

$L_X Y = [X, Y]$

Proof: Let f be an arbitrary differentiable map, then

$$L_X(Yf) = \lim_{t\to 0} \frac{\phi_t^*(Yf) - Yf}{t}$$
$$= \lim_{t\to 0} \frac{(\phi_t^* Y)(\phi_t^* f) - Yf}{t}$$

The Lie derivative of function and vector fields

$$= \lim_{t\to 0} \left[(\phi_t^* Y) \frac{\phi_t^* f - f}{t} + \frac{\phi_t^* Y - Y}{t} f \right]$$
$$= Y(L_X f) + (L_X Y) f,$$

But $L_X f = Xf$, thus

$$X(Y f) = L_X(Y f) = Y(Xf) + (L_X Y)f,$$

Then

$$(L_X Y)f = X(Y f) - Y(Xf) = [X, Y]f,$$

And hence

$$L_X Y = [X, Y].$$

It is the specialization of the Lie derivative to the case of Lie brackets of a vector field. Indeed, equals the Lie derivative.

■

-Special case: Lie derivative of the metric

Let g_{ij} be the metric of $\mathcal{M}^{2,1}$, we find

$$L_v g_{ij} = g_{ab},_c V^c + g_{ac} V^c,_b + g_{cb} V^c,_a$$

Then

$$L_v g_{ab} = g_{ac} V^c,_b + g_{cb} V^c,_a$$

If we rotate or boost then we always have

$$L_V \, g_{ab} = 0$$

The Lie derivative of the metric plays a key role in the theory of *Killing fields* see [[14]:120]. Which are generators of continuous isometries. A vector field is a Killing field, if the Lie derivative of the metric with respect to this field vanishes.

IX. Killing Vector Fields

Killing vector field or Killing vector ,"named after Wilhelm Killing, is a vector field on a Riemannian manifold (or pseudo-Riemannian manifold) that preserves the metric [17].

Definition The Killing vector field is a vector field which generates an isometry. Or Killing field is that vector V whose flow ϕ is one parameter group of isometry.

Properties of Killing field

Some important properties of Killing fields are stated below:

1. For any two vector fields, the linear combination between them is also Killing vector. i.e $aV + bW$ is KVF ,and $(a, b) \in \mathbb{R}$.

2. The Lie brackets of two Killing vector fields is also Killing vector field.

3. For a given Killing field , and geodesic with velocity vector u, the quantity $V_\mu u^\mu$ is constant along the geodesic [19]

Lemma [18] The vector V is Killing field if and only if $L_v g = 0$.

Proof: Suppose V is a Killing vector field and $\phi(u, x)$ be the one-parameter group of isometry. At each point x and each u. From the isometry part in section 3 above, we have the equation

$$g_{cd}(\phi(u,x)) \frac{\partial \phi^c(u,x)}{\partial x^a} \frac{\partial \phi^d(u,x)}{\partial x^b} = g_{ab}(x).$$

The right hand side is constant, so $^d/_{du}$ of the left hand side must vanish.

And we need to notice that :$\left(\frac{\partial}{\partial u}\right)\left(\frac{\partial \phi^c}{\partial x^a}\right) = \frac{\partial V^c}{\partial x^a}$.

So let's taking d/du both sides we have:

$$\partial_e g_{cd} V^e \frac{\partial \phi^c}{\partial x^a} \frac{\partial \phi^d}{\partial x^b} + g_{cd} \frac{\partial V^c}{\partial x^a} \frac{\partial \phi^d}{\partial x^b} + g_{cd} \frac{\partial \phi^c}{\partial x^a} \frac{\partial V^d}{\partial x^b} = 0.$$

Now, if we evaluate this at $\phi\,(0, x) = x$, then the derivative equal to

$$0 = V^e \partial_e g_{cd} \delta_a^c \delta_b^d + g_{cd} \frac{\partial V^c}{\partial x^a} \delta_b^d + g_{cd} \delta_a^c \frac{\partial V^d}{\partial x^b}$$

$$= V^e \partial_e g_{ab} + g_{cb} \frac{\partial V^c}{\partial x^a} + g_{ad} \frac{\partial V^d}{\partial x^b}$$

$= L_v g_{ab}$ As shown in previous section of Lie derivative

The backwards of this proof gives the converse.

Example In $\mathcal{M}^{2,1}$ by inspection , ∂_x, ∂_y and ∂_t are Killing vectors.

Also by inspection ∂_ϕ is a Killing vector. In Cartesian coordinates

$$(-y, x, 0)\partial_i = -y\partial_x + x\partial_y.$$

And the other $t\partial_x + x\,\partial_t$ and $t\,\partial_y + y\,\partial_t$.

X. Future Work

The future work expected is using rotation and two boosts in $\mathcal{M}^{2,1}$ in last example above and generate matrices of rotation in $\mathcal{M}^{2,1}$ which will be for future interest to rotate/boost an arbitrary curve around special axis. This will be some surfaces of rotations in Minkowski spaces.

XI. References

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