On the concept of Approximate Cofibration

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Abstract— In this article we study an important concept in the theory of fibration and cofibration, namely approximate cofibration (A-cofibration), which is the dual of the concept of approximate fibration [5, 10, 13], we give some examples. Following the known problems concerning the concept of cofibration as; the composition, the product, the pullback, the relation with retracts and so on, [1, 4, 6, 7 and 13], we give some similar results concerning A-cofibration.

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Index Terms—Fibration, cofibration

I. INTRODUCTION

There are two forms for fibration and cofibration as "Lifting Problem and Extension Problem, the familiar "Homotopy Extension Property" is special case of Extension Problem, in [5] give the formula of approximate Homotopy Lifting Property (A-HLP), which are generalize the concept of fibration, and hence holds for the larger set of maps. Poul and Matthey established a general method to produce co-fibrant approximations in the model category. In this work we study the some properties of approximate cofibration (A-cofibration) concept.

The word of mapping means continuous function, the word of space means topological space, and we replaced a l (Mixed fiber space), If \( X = X_1 \times I \rightarrow E \) such that \( \hat{H}(X, \alpha) \) = identity, \( f_1 \), \( f_2 \) are two fiber space and \( \alpha: X_2 \rightarrow X_1 \) such that \( f_1 \alpha = f_2 \) and \( \{ f_1, f_2 \} \) is a M-fiber space (Mixed fiber space). If \( X_1 = X_2 = X, \alpha = \text{identity} \) \( f_1 \), \( f_2 \) then \( \{ X, f, Y, \alpha \} \) is the usual fiber space.

2. Let \( \{ X, f, Y, \alpha \} \) be a M-fiber space such that \( f = f_1, f_2 \) then \( \{ X, f, Y, \alpha \} \) is the usual fiber space.

Definition(1-4) : Two maps \( f, k: X \rightarrow Y \), are said to be U-close, \( U \in \text{cov}(Y) \), provided that for each \( x \in X \), one element of \( U \) containing both \( f(x) \) and \( k(x) \). A map \( f: X \rightarrow Y \), is a near-homeomorphism if for any \( U \in \text{cov}(Y) \), there exist a homeomorphism of \( X \) onto \( Y \) which is U-close to \( f \).

Next, a maps \( f, k: X \rightarrow Y \), are U-homotopy, \( U \in \text{cov}(Y) \), iff it is a homotopic by a homotopy \( H: X \times I \rightarrow Y \), and \( H(1 \times I \times Y) \) contained in one element of \( U \).

Definition (1-5) : A proper map \( P: E \rightarrow B \), between locally compact ANR’s, has the approximate homotopy lifting property (A-HLP), w.r.t. a space \( X \), provided that \( \alpha \), given a \( U \in \text{cov}(B) \), a maps \( \kappa: X \rightarrow E \), and \( H: X \times I \rightarrow B \), such that \( P \circ \kappa = H \), there exist \( G: X \times I \rightarrow E \), such that \( G(1 \times 1 \times Y) \), and \( G(1 \times 1 \times Y) \) is U-close to \( H \); a map \( G \) is called U-lift of \( H \). Maps with the A-HLP, w.r.t. all spaces is A-fibration [5].

Definition (1-6) : A space \( X \) is said to be an ANR’s (absolute nbd retract), if for any space \( Y \) in which \( X \) can be embedded as a closed set there exist a nbd \( V \) of \( X \) in \( Y \) such that \( X \) is a retract of \( V \), (i.e. there exist \( r: V \rightarrow X \), such that \( r \circ j = 1_X \).

Next, a map \( j: A \rightarrow X \) is said to be have a homotopy extension property (HEP) w.r.t. a space \( Y \), Provided that given a map \( f: X \rightarrow Y \), a homotopy \( F: X \times I \rightarrow Y \), of \( f \), \( A \rightarrow Y \) such that \( f \circ j = F \). Then there exist a homotopy \( F: X \times I \rightarrow Y \). Such that the shown diagram, commutes, See [1, 6, 7, 10].

Definition (1-7) : A map \( j: A \rightarrow X \), is called a cofibration iff it has the (HEP) w.r.t. all spaces. Fibration and co-fibration, as well as various modifications of this notion, have some nice properties, which make them useful in studying both spaces and maps. There are several authors showed that some other classes of maps, e.g. cell-like maps, enjoy similar lifting properties [12]. This motivated Coram and Duvall [5], to define the notion of approximate fibrations.

This is a map \( P: E \rightarrow B \), between say compact ANR’s, has the approximate homotopy lifting property (A-HLP), w.r.t each \( X \in \mathfrak{A} \), where \( \mathfrak{A} \) is a given class of topological space , a map \( p \) is a cofibration w.r.t a given class of topological space , a map \( p \) is a cofibration w.r.t each \( X \in \mathfrak{A} \).

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An ordered pair of classes of maps $j: A\to X$, $p: Y\to B$, has relative lifting property (RLP) [8, 11], if for any diagram, a filler $f: X\to Y$, exist.

**Definition(1-12)** [11]: Given a class $\Sigma$ of maps; we call $j: A\to X$, a $\Sigma$-cofibration if $(j, p)$ has the (RLP) for all $p\in \Sigma$.

And $p: Y\to B$, is called a $\Sigma$-fibration if $(j, p)$ has the (RLP) for all $i\in \Sigma$.

**Definition(1-13)**: A proper map $j: X\to Y$, between locally compact ANR’s has the Approximate homotopy extension property (A-HEP) w.r.t. a space $Z$, provided that given $U\in \text{cov}(Z)$, a map $f: X\to Y$, a homotopy $h_\ast: X\to Z$, such that $f\circ j = h_\ast$, there exist a homotopy $f_\ast: Y\to Z$, such that $f_\ast = f$, and $f_\ast \circ j$ is U-close to $h_\ast$, where $f_\ast$ is said to be U-extended of $h_\ast$.

A map with (A-HEP) w. r. t. all spaces are called A-cofibration.

The above definition of course generalizes the usual cofibration (HEP), thus the A-cofibration (A-HEP) holds for a larger set of maps. As in [3, 5, 20], we called $(Y, X)$ an A-cofibered pair. It is clearly that, a cofibration is an A-cofibration, also the near-homeomorphism is A-cofibration.

**Proposition(1-14)**: Let $j_1: X\to Y$ is cofibration and $j_2: Y\to Z$ is A-cofibration, then $j_2 \circ j_1$ is A-cofibration.

**Proof:** Let $f_j: X\to E$ and $h: Z\to E$, be a given such that $h\circ j_2\circ j_1 = f_j\circ j_1$, since $j_1$ is cofibration, then there exist $f_j: Y\to E$, such that $f_\ast \circ j_1 = f_\ast\circ j_1\circ j_2$ and $f_\ast\circ j_1\circ j_2$ is A-cofibration, then there exist $h_2: Z\to E$, such that $h_2\circ j_2$ is U-close to $f_\ast (U\in \text{cov}(E))$, and $h_2 = h_\ast$.

Hence we have that $h_\ast\circ (j_2\circ j_1)$ is U-close to $f_j\circ j_1$, then $(j_2\circ j_1)$ is A-cofibration.

**Corollary(1-15)**: If $j: X\to Y$ and $i: Y\to Z$, be a maps such that $j$ is cofibration and $i\circ j$ is A-cofibration, then $i$ is A-cofibration.

**Proof:** Given a maps $f_i: Y\to E$ and $h: Z\to E$, such that $h\circ i = f_i$, since $(i\circ j)$ is A-cofibration then for any given $\kappa: X\to E$, such that $h\circ i\circ j = \kappa$, there exist $h_\ast: Z\to E$, such that $h\circ i\circ j$ is U-close to $\kappa$, “$U\in \text{cov}(E)$", and $h_\ast = h_\ast = h$, and since $j$ is cofibration then there exist $f_j: Y\to E$, Such that $f_\ast = f_\ast$ and $f_\ast\circ j = \kappa_1$. Hence we have $h\circ i$ is U-close to $f_j$, therefore $i$ is A-cofibration.
III. Maine results of A-cofibration related with A-retract

Firstly we will give the following terminology and notation; also we will introduce some definitions that we need.

If X is a subspace of a space Y such that the inclusion map \( X \subseteq Y \), is an A-cofibration, then the pair \( (Y, X) \) is called an A-cofibered pair or is said to possess the (A-HEP). A condition for \( (Y, X) \) to be an A-cofibered pair is the existence of approximate retraction (A-retracts), \( r_A: Y\times I \rightarrow \{(Y\times 0) \cup (X\times I)\} \).

**Definition(2-1):** The inclusion map \( j: X \rightarrow Y \) is called an approximate retract (A-retract), iff \( r_A \circ j \) is U-close to \( I_X \) for any \( U \in \text{cov}(X) \). If \( j \circ r_A \) is U-homotopic to \( I_Y \) U \( \in \text{cov}(Y) \), then \( j \) is an approximate deformation retracts (A-Dr). If \( j \circ r_A = U, I_Y \) rel \( X \), then \( j \) is an (A-SDr).

The first two theorems provide a tool for constructing examples of maps, which is the A-cofibration.

**Theorem(2-2):** If \( j : X \rightarrow Y \) is an A-cofibration then \( j \) is a near-homeomorphism.

**Proof:** Conceder the following diagram;

Let \( j : X \rightarrow Y \) be an A-cofibration, Consider Z= \( \{(Y\times 0) \cup (X\times I)\} \), is the quotient space of topological sum obtained by identifying \( (x, 0) \) with \( (j(x), 0) \).

Let \( q \) be the quotient map \( q : \{(Y\times 0) \cup (X\times I)\} \rightarrow Z \), that is a map \( h : Z \rightarrow Y\times I \), define as \( h \circ q (y, 0) = (y, 0), y \in Y \) and \( h \circ q (x, t) = (j(x), t) \).

Let \( f : Y \rightarrow Z \) and \( F : X \times I \rightarrow Z \), such that \( f(y) = q(y, 0) \) and \( F(x, t) = q(x, t) \). Since \( j \) is an A-cofibration, then for any \( U \in \text{cov}(Z) \), there exist \( F : Y\times I \rightarrow Z \), such that \( F(y, 0) = q(y, 0) \) and \( F(j(x), t) \rightarrow (x, t) \). Hence \( F \circ h \) is U-close to \( I_Z \).

Conversely:

**Lemma(2-2):** If \( (Y, X) \) is an A-cofibered pair, then \( \{(Y\times 0) \cup (X\times I)\} \) is the quotient space of topological sum obtained by identifying \( (x, 0) \) with \( (j(x), 0) \).

**Corollary(2-4):** If \( (Y, X) \) is an A-cofibered pair, then \( \{(Y\times 0) \cup (X\times I)\} \) is an A-cofibered pair, so \( \{Y\times 0\} \cup (X\times I) \) is a (A-SDr), of \( Y\times I \).

**Proof:** The U-homotopy between \( j \circ r_A \) and \( I_Y \) will be given as;

\[ H_U(y, t) = \begin{cases} (P_1 \circ r_A(y, (1 - t) \times)) & \text{if } t \leq \varphi(y) \\ (P_1 \circ r_A(y, t + \varphi(y))) & \text{if } t > \varphi(y) \end{cases} \]

**Remark:**

If \( \varphi(y) \leq 1 \), then \( H_U(y, (y, \varphi)) \in \text{Hom} \{(Y, t) \times (\varphi(y), I)\} \subseteq X \), thus replacing \( H_U(y, t) \) by \( H_U(y, t \cap (\varphi(y))) \), We have the following result.

**Corollary(2-6):** If \( (Y, X) \) is A-cofibration, so is \( (Y, X) \). The following lemma is generalized of (1-7), in [3], and we needed in the last section, that we well give the proof of it.

**Lemma(2-7):** If \( X \subseteq Y \) is A-cofibration, then the inclusion map \( j : X \rightarrow Y \) is a U-homotopy equivalence.
Proof:
Since X is (A-Dr) of Y, then there exist D_A: Y×I→Y, such that d_1= A-is retract of Y onto X (r_A: Y→X), then r_A ◦ j is the identity on X; then j is a homotopy equivalence.

Lemma(2-8): Suppose that P: E→B is A-fibration, with X is an (A-SDr) of Y, and that there exist a map φ: Y→I, such that X= φ^(-1)(0), then a U-commutative diagram;

\[
\begin{array}{ccc}
\emptyset & \rightarrow & E \\
\downarrow & & \downarrow \\
Y & \rightarrow & B \\
\end{array}
\]

My be filled in with a map f: Y→E, such that P ◦ f is U-close to f, and f ◦ j= f^0, f is unique up to a U-homotopy; (U∈cov(B)).

Proof: By hypothesis, there exist (A-SDr), D_A: j ◦ r_1= i_Y, rel X. Define D_A: Y×I→Y, by;

D_A(y, t) = \{ D_A(y, t) \mid \phi(y), t ≤ \phi(y) \}

Since P is A-fibration, there exist U-homotopy, F_U: Y×I→E, such that P ◦ F_U is U-close to f. Since P ◦ F_U= f, we defined f as f(y)= F_U(y, \phi(y)). If f: Y→E, is such that f ◦ j= f^0, then f= f ◦ j ◦ r = f^0 ◦ r, rel X.

Theorem(2-9): Suppose that P: E→B is A-fibration and j: X→Y is a cofibration, which X is closed, then any U-commutative diagram;

\[
\begin{array}{ccc}
([Y×0] \cup (X×I)) & \rightarrow & E \\
\downarrow \left(\begin{array}{c}j \\
F \end{array}\right) & & \downarrow \left(\begin{array}{c}p \\
\end{array}\right) \\
Y×I & \rightarrow & B \\
\end{array}
\]

May be filled in with a homotopy F: Y×I→E, such that P ◦ F is U-close to f, for any U∈cov(B), and F \mid ([Y×0] \cup (X×I))= f.

Proof: By corollary (4-3), ([Y×0] \cup (X×I)) is (A-SDr) of Y×I; That is D_A: j ◦ r_1= i_Y, rel ([Y×0] \cup (X×I)). And by lemma (5-3), there exist a function ψ: Y→I, such that X= ψ^(-1)(0). Define φ: Y×I→I, by φ(y, t) = tψ(y), then [Y×0] \cup (X×I)= φ^(-1)(0). And hence the theorem follows from (1-4).

Definition(2-10): Let j: X→Y and P: E→B are maps, a map pair f=(f^0, f^1): j→P, is a pair of maps f^0: X→E and f^1: Y→B, such that the diagram is U-commutes, U∈cov(B).

And a map ġ: Y→E defines a map pair Q(ġ)= (ġ ◦ j, P ◦ ġ): j→P, ġ is called a U-lifting of the pair Q(ġ).

Theorem(2-11): Let j: X→Y, be map with j(X) closed; then j is A-cofibration and U-homotopy equivalence, iff a map pair f: j→P, with P: E→B is A-fibration, has U-lifting.

Proof: Suppose that j is A-cofibration and since P be a given A-fibration then the first direction is just lemma (1-4).

Conversely; since \exists j: Z′→Z is A-fibration for any space Z, hence we consider the U-commutative diagram;

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow \left(\begin{array}{c}j \\
F \end{array}\right) & & \downarrow \left(\begin{array}{c}\hat{\phi} \\
G \end{array}\right) \\
Y & \rightarrow & Z \\
\end{array}
\]

Where F is U-lifting of its map pair. And so j: X→Y must be A-cofibration. Next, we may assume j is an inclusion map, and since X→{p}, is A-fibration ({p} denotes a one-point space); then an A-retraction r_A: Y→X is obtained as U-lifting of the map pair of the following diagram;

\[
\begin{array}{ccc}
X & \rightarrow & X \\
\downarrow \left(\begin{array}{c}j \\
F \end{array}\right) & & \downarrow \left(\begin{array}{c}\phi \\
\end{array}\right) \\
Y & \rightarrow & Y \times Y \\
\end{array}
\]

With f^0(0)= y, f^1(0)= (y, r(y)), has U-lifting ţ: Y→Y^2 associate to (A-SDr) of Y to X, and the complete of proof follows from (3-7).

Corollary(2-12): In the above theorem if the equivalence holds, then the U-lifting of f is unique up to a U-homotopy relative to j(X).

The proof of the following theorem is a similar fashion:

Theorem(2-13): For a map P: E→B, the map pair f: j→P, with j is a closed A-cofibration, has U-lifting iff P is an A-fibration and U-homotopy equivalence.

IV. INDUCED A-COFACTORIONS:

Let j: X→Y, f: X→E, be a maps and Y^ι = E ↝ Y, be the cofibers sum of Y and E, which is the set of all equivalents classes of topological sum under the equivalence relation generated by [e•y ⇔ ∃ x∈X; ε = f(x), y = j(x)]. Let q: E→Y ↝ E ↝ Y is the identification map.
Define \( f^j : Y \to Y^j \), \( j : E \to Y \), as the composition of \( q \) with the inclusions of \( E \) and \( Y \) into \( E \times Y \) res., then \( j^j \) is called the pushout of \( j \) by \( f \).

**Theorem (3-1):** The pushout of an A-cofibration is also an A-cofibration.

**Proof:** Let \( j : X \to Y \) be an A-cofibration, and \( f : X \to E \) be a map. Let \( j^j : E \times Y \to E \times Y \) be the pushout of \( j \) by \( f \); so for any space \( Z \), let \( \kappa : Y^j \to Z \) and \( H : E \times I \to Z \), such that \( \kappa \circ f^j = H \), and we have that \( \kappa \circ f^j : Y \to Z \) and \( H(f \times 1) : X \times I \to Z \), such that \( \kappa \circ f^j = H \circ (f \times 1) \); since \( j \) is an A-cofibration, there exist \( F : Y^I \to Z \), such that \( \kappa \circ f^j = F \circ \eta \), and \( F \circ \eta = H \circ (f \times I) \), which \( U \in \text{cov}(Z) \). Hence define \( F^j : Y^j \times I \to Z \), by \( F^j (j^j(y), t) \leftarrow F(y, t), \) and \( F^j(j^j(y), t) \leftarrow H(e, t) \), then \( F^j(j^j \times I) \) is U-close to \( H \) and \( F^j(j^j(y), 0) = F(y, 0) = \kappa \circ j^j(y) = \kappa (\eta(y)) \), also \( F^j(j^j(y), t) = H(e, t) = \kappa \circ j^j(e) = \kappa (\eta(e)) \), which \( (q : E \times Y \to E \cup Y) \).

**Theorem (3-2):** If \( j : X \to Y \) and \( i : X^j \to Y^j \), are A-cofibration with \( X \) closed in \( Y \), then \( \{Y, X \times (Y^j, X^j) \} = (Y \times Y^j, Y \times X^j \cup X \times Y^j) \) is also A-cofibration.

**Proof:** Let \( \psi : Y \to I \) and \( H_0 : Y^j \times I \to Y \), be as described in lemma (2-3),

Let \( \psi \) and \( G_{11} \), be the corresponding maps for \((Y^j, X^j)\); define \( \eta : Y \times Y^j \to I \) and \( F_{11} : Y \times Y^j \times I \to Y \times Y^j \), by \( F_{11}(y, y', t) \leftarrow H_0(y, t \wedge \psi(y')) \), \( G_{11}(y, y', t \wedge \psi(y')) \), and \( \eta(y, y') \leftarrow \eta(y) \wedge \psi(y') \). Then \( F_{11}(y, y', t) \leftarrow (Y \times X^j \cup X \times Y^j) \subset \{y \wedge (0) \text{ or } t \geq 0 \} \text{ or } y, y' \in X \times Y^j \). Now suppose \( t > \eta(y, y') \); then \( \eta(y, y') \leftarrow \eta(y) \wedge \psi(y') \), and \( \eta(y, y') \leftarrow \eta(y) \wedge \psi(y') \), in which \( t \wedge \psi(y') \leftarrow \eta(y) \wedge \psi(y') \), and \( F_{11}(y, y', t \wedge \psi(y')) \leftarrow \eta(y) \wedge \psi(y') \), whenever \( t > \eta(y, y') \), and therefore from lemma (5-3), that \((Y \times Y^j, Y \times X^j \cup X \times Y^j) \) is A-cofibration.

**Theorem (3-3):** Suppose that \( X \times Y \), that there exists a continuous function \( \phi : Y \to I \), with \( X \subset \phi^{-1}(0) \) and that there exist a point \( y_0 \in Y \times X \), such that \( \phi(y_0) \neq 0 \); also if \((Y^j, X^j) \) is a pair such that \((Y \times Y^j, Y \times X^j \cup X \times Y^j) \), is A-cofibration, then we have that \((Y \times Y^j, X^j) \) itself is A-cofibration.

**Proof:** Let \( \eta : Y \times Y^j \to I \) and \( F_{11} : Y \times Y^j \times I \to Y \times Y^j \), be as described in (3-2),

We may obviously assume that \( \phi(y_0) = 1 \). Define \( G_{11} : Y \times I \to Y \times Y^j \) and \( \psi : Y \to I \), by: \( \psi(y') \leftarrow \min \{ (\eta(y_0, y'), 1) \text{ or } \phi \circ F_{11}(y_0, y', t) \} \), and \( G_{11}(y', t) = [P_1 \circ F_{11}(y_0, y', t)] \), which will be satisfy the condition of (2-5).