Solving fuzzy fractional Riccati differential equations by the variational iteration method

Ekhtiar Khodadadi, Mesut Karabacak, Ercan Çelik

Abstract— **This paper deals with the solutions of fuzzy fractional differential equations (FFDEs) under Caputo H-differentiability by variational iteration method. The variational iteration method has been applied in solving fuzzy fractional Riccati differential equations with fuzzy initial conditions. This method is illustrated by solving two examples.**

Index Terms— **variational iteration method, fuzzy number, fuzzy fractional Riccati differential equation, fuzzy initial value problem**

I. Introduction

Recently, the subject of fuzzy differential equations (FDEs) has been quickly increasing. Initially, the term of the fuzzy derivative was defined by Chang and Zadeh [1]; it was pursued by Dubois and Prade [2], who made use of the extension principle in their study. Park, Kwan and Jeong [3] have discussed other methods to see if it satisfies the differential equation with fuzzy initial conditions; and the second method was the reverse of the first method, in that they initially solved the fuzzy initial value problem and they checked to see if it defined a fuzzy function. Fuzzy fractional differential equations play progressively significant roles in the modeling of science and engineering problems such as civil engineering, population models and in modeling hydraulic. It has been demonstrated that, convenient results than analytical models with integer derivatives. In this paper, the approximate solution of fuzzy fractional order differential equation will be discussed, in which fractional differential equation could be regarded as an essential type of differential equations, where the different integration that appears in the equation is of non-integer order. Here we consider the following nonlinear fuzzy fractional Riccati differential equation:

$$ {}^{C}D_{0^+}^{\alpha} y(t) = A(t) + B(t)y + C(t)y^2, \quad t > 0, \ m-1 < \alpha \le m, \tag{1.1}$$

with fuzzy initial conditions:

$$y^{(j)}(0) = \tilde{c}_j, \ j = 0,1,\dots,m-1, \ m \in N,$$

where $A(t)$, $B(t)$ and $C(t)$ are given functions, $\tilde{c}_j$, $j = 0,1,\dots,m-1$, $m \in N$, are arbitrary fuzzy number and α is an order of the fractional derivative.

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The most commonly utilized definitions are the fuzzy Riemann-Liouville and Caputo definitions. Each definition makes use of fuzzy Riemann-Liouville fractional derivatives and integration of integer order.
The distinction between the two definitions is in the order of evaluation order. Fuzzy Riemann-Liouville fractional integration of order α is defined as

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi)\, d\xi, \quad \alpha > 0, \ t > 0, \tag{1.2}$$

The following two equations define fuzzy Riemann-Liouville and fuzzy Caputo fractional derivative of order α, respectively:

$$ {}^{RL}D_{0^+}^{\alpha} f(t) = \frac{d^m}{dt^m}\left(I_t^{m-\alpha} f(t)\right) \quad m-1 < \alpha \le m \text{ and } m \in \mathbb{N} \tag{1.3}$$

$$ {}^{C}D_{0^+}^{\alpha} f(t) = I_t^{m-\alpha}\left(\frac{d^m}{dt^m} f(t)\right), \qquad m-1 < \alpha \le m \text{ and } m \in \mathbb{N} \tag{1.4}$$

We would rather use the fuzzy Caputo fractional derivative. Because it permits traditional initial and boundary conditions to be included in the formulation of the problem, For more details on the geometric and physical interpretation for fuzzy fractional derivatives, see [4, 5, 6, 7].

II. Preliminaries and Notations

Definition 2.1. A fuzzy number is a fuzzy set $u: \mathbb{R} \longrightarrow \mathbb{E} = [0,1]$ which satisfies

1) u is upper semicontinuous.
2) $u(t) = 0$ outside some interval $[c,d]$.
3) There are real numbers a, b: $c \le a \le b \le d$ for which
   i. $u(t)$ is monotonic increasing on $[c,a]$.
   ii. $u(t)$ is monotonic increasing on $[b,d]$.
   iii. $u(t) = 1, a \le t \le b$.

The set of all fuzzy numbers (as given by Definition 2.1) is denoted by $\mathbb{E}^1$. An alternative definition or parametric form of a fuzzy number which yields the same $\mathbb{E}^1$ is given by Kaleva [8].

Definition 2.2. A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r)$, $0 \le r \le 1$, which satisfy the following requirements [9]:

1) $\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0,1]$ and right continuous at 0,
2) $\overline{u}(r)$ is a bounded non-decreasing left continuous function in $(0,1]$ and right continuous at 0,
3) $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

Definition 2.3. A fuzzy number u is called a triangular fuzzy number (TFN) if its member membership function u is given by

$$u(t)=\begin{cases}0, & t<a\\ \frac{t-a}{b-a}, & a\le t\le b\\ \frac{t-c}{b-c}, & b\le t\le c\\ 0, & c<t\end{cases},$$

The TFN is denoted by the triplet $u=(a,b,c)$. Further the r-cut of the TFN $u=(a,b,c)$ is the closed interval $u^r=[\underline{u},\overline{u}]=[(b-a)r+a,-(c-a)r+c], r\in(0,1]$, [10].

Definition 2.4. Fractional derivative of compounded functions [11] is defined as

$$d^\alpha f\cong\Gamma(1+\alpha)df,\quad 0<\alpha<1.$$

Definition 2.5. The integral with respect to $(dt)^\alpha$ [11] is defined as the solution of the fractional differential equation

$$dy\cong f(t)(dt)^\alpha, t\ge 0,\ y(0)=0,\ 0<\alpha<1 \qquad (2.1).$$

Lemma 2.1. Let $f(t)$ denote a continuous function [11] then the solution of the Eq. (2.1) is defined as

$$y=\int_0^t f(\xi)(d\xi)^\alpha=\alpha\int_0^t(t-\xi)^{\alpha-1}f(\xi)\,d\xi,\ \ 0<\alpha\le 1, \qquad (2.2)$$

For example $f(t)=t^\gamma$ in Eq. (2.2) we can write,

$$\int_0^t\xi^\gamma(d\xi)^\alpha=\frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma},\ \ 0<\alpha\le 1. \qquad (2.3)$$

III. Properties of Modified Fuzzy Riemann-Liouville Derivative

Comparing with the classical fuzzy Caputo derivative, the definition of modified fuzzy Riemann-Liouville derivative is not required to satisfy higher integer-order derivative than α. Also, α^{th} derivative of a constant is zero. Now we show some properties of the fractional derivative. Suppose that $f:\mathbb{R}\longrightarrow\mathbb{R}$, $t\longrightarrow f(t)$ denote a continuous (but not necessarily differentiable) function in the interval $[0,1]$. Through the fuzzy fractional Riemann-Liouville integral

$$\left[(I_{0^+}^\alpha f)(t)\right]^r=\left[\left(I_{0^+}^\alpha\underline{f}\right)(t;r),\left(I_{0^+}^\alpha\overline{f}\right)(t;r)\right],\ \ 0\le r\le 1, \qquad (3.1)$$

where

$$\left(I_{0^+}^\alpha\underline{f}\right)(t;r)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\underline{f}(\xi;r)d\xi}{(t-\xi)^{1-\alpha}},\ \ t>0,\ \ 0<\alpha\le 1, \qquad (3.2)$$

and

$$\left(I_{0^+}^\alpha\overline{f}\right)(t;r)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\overline{f}(\xi;r)d\xi}{(t-\xi)^{1-\alpha}},\ \ t>0,\ \ 0<\alpha\le 1, \qquad (3.3)$$

the modified fuzzy Riemann-Liouville derivative is defined as [6]

$$\left[{}^{RL}D_{0^+}^\alpha f(t)\right]^r=\left[{}^{RL}D_{0^+}^\alpha\underline{f}(t;r),{}^{RL}D_{0^+}^\alpha\overline{f}(t;r)\right],\ \ 0\le r\le 1, \qquad (3.4)$$

where

$${}^{RL}D_{0^+}^\alpha\underline{f}(t;r)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{\underline{f}(\xi;r)d\xi}{(t-\xi)^\alpha} \qquad (3.5)$$

and

$${}^{RL}D_{0^+}^\alpha\overline{f}(t;r)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{\overline{f}(\xi;r)d\xi}{(t-\xi)^\alpha} \qquad (3.6)$$

In the next sections, we will use the integration with respect to $(dt)^\alpha$ (**Lemma 2.1**),

$$\left[I_{0^+}^\alpha f(t)\right]^r=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}[f(\xi)]^r d\xi=\frac{1}{\Gamma(\alpha+1)}\int_0^t[f(\xi)]^r(d\xi)^\alpha,0<\alpha\le 1. \qquad (3.7)$$

IV. Analysis of the Variational Iteration Method

We consider the fuzzy fractional Riccati differential equation (1.1), According to the variational iteration method [12], we construct a correction functional for (1.1) which reads

$$\begin{cases}\underline{y}_{n+1}=\underline{y}_n+I_x^\alpha\underline{\lambda}(\xi)\left[\frac{d^\alpha\underline{y}_n}{d\xi^\alpha}-A(\xi)-B(\xi)\underline{y}_n-C(\xi)\underline{y}_n^2\right]\\ \overline{y}_{n+1}=\overline{y}_n+I_x^\alpha\overline{\lambda}(\xi)\left[\frac{d^\alpha\overline{y}_n}{d\xi^\alpha}-A(\xi)-B(\xi)\overline{y}_n-C(\xi)\overline{y}_n^2\right]\end{cases}, \qquad (4.1)$$

To identify the multiplier, we approximately write (4.1) in the form

$$\begin{cases}\underline{y}_{n+1}=\underline{y}_n+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}\lambda(\xi)\,d\xi\\ \quad\left(\frac{d^\alpha\underline{y}_n}{d\xi^\alpha}-A(\xi)-B(\xi)\underline{y}_n-C(\xi)\underline{y}_n^2\right)\\ \overline{y}_{n+1}=\overline{y}_n+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}\lambda(\xi)\,d\xi\\ \quad\left(\frac{d^\alpha\overline{y}_n}{d\xi^\alpha}-A(\xi)-B(\xi)\overline{y}_n-C(\xi)\overline{y}_n^2\right)\end{cases}, \qquad (4.2)$$

Using Eq. (2.2), we obtain a new correction functional

$$\begin{cases}\underline{y}_{n+1}=\underline{y}_n+\frac{1}{\Gamma(\alpha+1)}\int_0^t\underline{\lambda}(\xi)\left[\frac{d^\alpha\underline{y}_n}{d\xi^\alpha}-A(\xi)-B(\xi)\underline{y}_n-C(\xi)\underline{y}_n^2\right](d\xi)^\alpha\\ \overline{y}_{n+1}=\overline{y}_n+\frac{1}{\Gamma(\alpha+1)}\int_0^t\overline{\lambda}(\xi)\left[\frac{d^\alpha\overline{y}_n}{d\xi^\alpha}-A(\xi)-B(\xi)\overline{y}_n-C(\xi)\overline{y}_n^2\right](d\xi)^\alpha\end{cases}, \qquad (4.3)$$

where $\underline{\lambda}$ and $\overline{\lambda}$ are the so-called general Lagrange multiplier [13]. It is obvious that the successive approximation $\underline{y}_j$ and $\overline{y}_j$, $j\ge 0$ can be established by determining $\underline{\lambda}$ and $\overline{\lambda}$, general Lagrange multipliers, which can be identified optimally via the variational theory [14,15]. The function $\tilde{y}_n$ is a limited variation, which means $\delta\tilde{y}_n=0$. Thus, we first write the Lagrange multiplier $\underline{\lambda}$ and $\overline{\lambda}$ that will be identified by way of integration by parts. The successive approximations $\underline{y}_{n+1}(t;r)\ge 0$ of the solution $\underline{y}(t;r)$ and $\overline{y}_{n+1}(t;r)\ge 0$ of the solution $\overline{y}(t;r)$ will be easily obtained upon using the Lagrange multipliers gotten by using any selective functions $\underline{y}_0(t;r)$ and $\overline{y}_0(t;r)$. Hereby, the fuzzy solutions are written by using the limits:

$$\begin{cases}\underline{y}(t;r)=lim_{n\to\infty}\underline{y}_n(t;r)\\ \overline{y}(t;r)=lim_{n\to\infty}\overline{y}_n(t;r)\end{cases} \qquad (4.4)$$

V. Applications and Numerical Results

In this section, to show the efficiency of this method, we present two illustrative examples.

Example 1. Consider the following fuzzy fractional Riccati differential equation:

$$ {}^{C}D_{0^+}^{\alpha} y(t) = 2y(t) - y^2(t) + 1, \quad t > 0, \ \ 0 < \alpha \le 1, \tag{5.1}$$

subject to the fuzzy initial condition

$y(0) = \tilde{0} = [r-1, 1+r]$.

The exact solution of (5.1) is

$y(t) = 1 + \sqrt{2}\, tanh\left(\sqrt{2}t + \left(\frac{1}{2}\right) log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right)$, when $\alpha = 1$.

Taylor expansion of $y(t)$ around $t = 0$ gives

$$y(t) = t + t^2 - \frac{t^3}{3} - \frac{t^4}{3} - \frac{7t^5}{15} - \frac{7t^6}{45} + \frac{53t^7}{315} + \frac{71t^8}{315} + \cdots.$$

Now, the correction functional for (5.1) form as below

$$\begin{cases} \underline{y}_{n+1}(t;r) = \underline{y}_n(t;r) + \frac{1}{\Gamma(\alpha+1)} \\ \int_0^t \underline{\lambda}(\xi)\left(\frac{d^{\alpha}\underline{y}_n(\xi;r)}{d\xi^{\alpha}} - 2\underline{y}_n(\xi;r) + \underline{y}_n^{\,2}(\xi;r) - 1\right)(d\xi)^{\alpha} \\ \overline{y}_{n+1}(t;r) = \overline{y}_n(t;r) + \frac{1}{\Gamma(\alpha+1)} \\ \int_0^t \overline{\lambda}(\xi)\left(\frac{d^{\alpha}\overline{y}_n(\xi;r)}{d\xi^{\alpha}} - 2\overline{y}_n(\xi;r) + \overline{y}_n^{\,2}(\xi;r) - 1\right)(d\xi)^{\alpha} \end{cases}, \tag{5.2}$$

where $\frac{\partial^{\alpha}[y_n(\xi)]^r}{\partial\xi^{\alpha}} = {}^{C}D_{0^+}^{\alpha}[y_n(\xi)]^r$. This yields the stationary conditions $\underline{\lambda}'(\xi) = 0$, $1 + \underline{\lambda}(\xi) = 0$ and $\overline{\lambda}'(\xi) = 0$, $1 + \overline{\lambda}(\xi) = 0$ which gives

$$\underline{\lambda}(\xi) = \overline{\lambda}(\xi) = -1,$$

Using this Lagrangian multiplier in (5.2), the iteration formula is shaped below

$$\begin{cases} \underline{y}_{n+1}(t;r) = \underline{y}_n(t;r) - \frac{1}{\Gamma(\alpha+1)}\int_0^t \left(\frac{d^{\alpha}\underline{y}_n(\xi;r)}{d\xi^{\alpha}} - 2\underline{y}_n(\xi;r) + \underline{y}_n^{\,2}(\xi;r) - 1\right)(d\xi)^{\alpha} \\ \overline{y}_{n+1}(t;r) = \overline{y}_n(t;r) - \frac{1}{\Gamma(\alpha+1)}\int_0^t \left(\frac{d^{\alpha}\overline{y}_n(\xi;r)}{d\xi^{\alpha}} - 2\overline{y}_n(\xi;r) + \overline{y}_n^{\,2}(\xi;r) - 1\right)(d\xi)^{\alpha} \end{cases} \tag{5.3}$$

Beginning with

$\underline{y}_0(t;r) = (r-1)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$,

$\overline{y}_0(t;r) = (1-r)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$,

by the iteration formulations (5.3), we get directly the other components as

$\underline{y}_1(t;r) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} + 2(r-1)\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha} - (r-1)^2\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}t^{3\alpha}$,

$\overline{y}_1(t;r) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} + 2(1-r)\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha} - (1-r)^2\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}t^{3\alpha}$,

$$\begin{aligned} \underline{y}_2(t;r) &= \frac{t^{\alpha}}{\Gamma(\alpha+1)} + 2\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha} + 4(r-1)\frac{1}{\Gamma(3\alpha+1)}t^{3\alpha} \\ &- \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}t^{3\alpha} \\ &- 2(r-1)^2\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(4\alpha+1)}t^{4\alpha} \\ &- 4(r-1)\frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}t^{4\alpha} \\ &+ 2(r-1)^2\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha} - \\ & 4(r-1)^2\frac{\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(5\alpha+1)}t^{5\alpha} \\ &+ 4(r-1)^3\frac{\Gamma(5\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)\Gamma(6\alpha+1)}t^{6\alpha} - \\ & (r-1)^4\frac{(\Gamma(2\alpha+1))^2\Gamma(6\alpha+1)}{(\Gamma(\alpha+1))^4(\Gamma(3\alpha+1))^2\Gamma(7\alpha+1)}t^{7\alpha}, \end{aligned}$$

$$\begin{aligned} \overline{y}_2(t;r) &= \frac{t^{\alpha}}{\Gamma(\alpha+1)} + 2\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha} \\ &+ 4(1-r)\frac{1}{\Gamma(3\alpha+1)}t^{3\alpha} \\ &- \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}t^{3\alpha} \\ &- 2(1-r)^2\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(4\alpha+1)}t^{4\alpha} \\ &- 4(1 \\ &- r)\frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}t^{4\alpha} \\ &+2(1-r)^2\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha} - \\ & 4(1-r)^2\frac{\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(5\alpha+1)}t^{5\alpha} \\ &+ 4(1-r)^3\frac{\Gamma(5\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)\Gamma(6\alpha+1)}t^{6\alpha} - \\ & (1-r)^4\frac{(\Gamma(2\alpha+1))^2\Gamma(6\alpha+1)}{(\Gamma(\alpha+1))^4(\Gamma(3\alpha+1))^2\Gamma(7\alpha+1)}t^{7\alpha}, \end{aligned}$$

$\vdots$

and so on. The *n*th approximate fuzzy solutions of the method approximates to the exact series solution [4]. So, we approximate fuzzy solutions

$$\underline{y}(t;r) = lim_{n\to\infty}\, \underline{y}_n(t;r) \text{ and } \overline{y}(t;r) = lim_{n\to\infty}\, \overline{y}_n(t;r).$$

In Figure 2, approximate fuzzy solution

$[y(t)]^r = \left[\underline{y}_3(t;r), \overline{y}_3(t;r)\right]$ using VIM is plotted for $\alpha = 0.5$

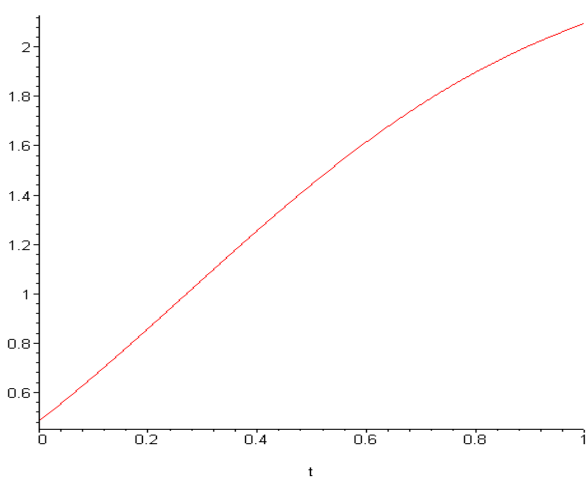

Figure 1. Exact solution.

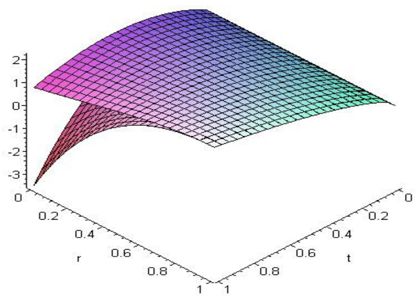

Figure 2. The approximate fuzzy solutions to the VIM

Example 2. (Fuzzy fractional nuclear decay equation). Consider the following fuzzy fractional Riccati differential equation

$$\begin{cases} ({}^{C}D_{0^+}^{\alpha}y)(t) = -y(t), \quad 0 < x, 0 < \alpha \leq 1 \\ y(0) = \tilde{1} = [0.5 + 0.5r, 1.5 - 0.5r] \end{cases} \tag{5.4}$$

where, $y(t)$ is the count of total radionuclides existing in any radioactive. The exact solution of (5.4) is

$$y(t) = \sum_{k=0}^{m-1} t^k E_{\alpha,k+1}(-t^\alpha) y^{(k)}(0).$$

Firstly we write the iteration form as below,

$$\begin{cases} \underline{y}_{n+1}(t;r) = \underline{y}_n(t;r) + \frac{1}{\Gamma(\alpha+1)} \\ \int_0^t \lambda(\xi)\left(\frac{d^\alpha \underline{y}_n(\xi;r)}{d\xi^\alpha} + \underline{y}_n(\xi;r)\right)(d\xi)^\alpha \\ \overline{y}_{n+1}(t;r) = \overline{y}_n(t;r) + \frac{1}{\Gamma(\alpha+1)} \\ \int_0^x \lambda(\xi)\left(\frac{d^\alpha \overline{y}_n(\xi;r)}{d\xi^\alpha} + 2\overline{y}_n(\xi;r)\right)(d\xi)^\alpha \end{cases},$$

where $\frac{\partial^\alpha [y_n(\xi)]^r}{\partial \xi^\alpha} = {}^{C}D_{0^+}^{\alpha}[y_n(\xi)]^r$ and $\lambda(\xi)$ is unknown but it can be determined.

$$\delta \underline{y}_{n+1}(t;r) = \delta \underline{y}_n(t;r) + \frac{\delta}{\Gamma(\alpha+1)} \int_0^t \lambda(\xi)\left(\frac{d^\alpha \underline{y}_n(\xi;r)}{d\xi^\alpha} + \underline{y}_n(\xi;r)\right)(d\xi)^\alpha =$$
$$(1 + \lambda|_{\xi=x})\delta \underline{y}_n(x;r) - \frac{1}{\Gamma(\alpha+1)} \int_0^x \left(\lambda_\xi^{(\alpha)} + \lambda\right) \delta \underline{y}_n(\xi;r)(d\xi)^\alpha,$$
$$\delta \overline{y}_{n+1}(t;r) = \delta \overline{y}_n(t;r) + \frac{\delta}{\Gamma(\alpha+1)} \int_0^t \lambda(\xi)\left(\frac{d^\alpha \overline{y}_n(\xi;r)}{d\xi^\alpha} + \overline{y}_n(\xi;r)\right)(d\xi)^\alpha =$$
$$(1 + \lambda|_{\xi=x})\delta \overline{y}_n(x;r) - \frac{1}{\Gamma(\alpha+1)} \int_0^x \left(\lambda_\xi^{(\alpha)} + \lambda\right) \delta \overline{y}_n(\xi;r)(d\xi)^\alpha.$$

Note that $\delta \underline{y}_n(0;r) = 0$ and $\delta \overline{y}_n(0;r) = 0$. $\lambda(\xi)$ must satisfy

$$1 + \lambda|_{\xi=t} \text{ and } \lambda_\xi^{(\alpha)} - \lambda = 0,$$

As a result, $\lambda(\xi)$ can be identified explicitly

$$\lambda(\xi) = -E_{\alpha,1}((\xi - t)^\alpha) \tag{5.5}$$

where $E_{\alpha,1}((\xi - t)^\alpha)$ is defined by the classical Mittag-Leffler function

$$E_{\alpha,1}((\xi - t)^\alpha) = E_{\nu,\mu}(z) = \sum_{n=0}^{\infty} \frac{((\xi-t)^\alpha)^n}{\Gamma(\alpha n+1)}.$$

Thus, the iteration formulae for Eq. (5.4) can be written as

$$\begin{cases} \underline{y}_{n+1}(t;r) = \underline{y}_n(t;r) - \frac{1}{\Gamma(\alpha+1)} \\ \int_0^t E_{\alpha,1}((\xi - x)^\alpha)\left(\frac{d^\alpha \underline{y}_n(\xi;r)}{d\xi^\alpha} + \underline{y}_n(\xi;r)\right)(d\xi)^\alpha \\ \overline{y}_{n+1}(t;r) = \overline{y}_n(t;r) - \frac{1}{\Gamma(\alpha+1)} \\ \int_0^t E_{\alpha,1}((\xi - x)^\alpha)\left(\frac{d^\alpha \overline{y}_n(\xi;r)}{d\xi^\alpha} + \overline{y}_n(\xi;r)\right)(d\xi)^\alpha \end{cases}.$$

On the other hand, if $\underline{y}_n(t;r)$ and $\overline{y}_n(t;r)$ is handled as a limited variation in Eq. (5.5), similarly, the Lagrange multiplier can be identified by

$$1 + \lambda|_{\xi=t} \text{ and } \lambda_\xi^{(\alpha)} = 0.$$

As a result, we can derive the generalized multiplier

$$\lambda(\xi) = -1,$$

we can get the iteration form

$$\begin{cases} \underline{y}_{n+1}(t;r) = \underline{y}_n(t;r) - \frac{1}{\Gamma(\alpha+1)} \int_0^t \left(\frac{d^\alpha \underline{y}_n(\xi;r)}{d\xi^\alpha} + \underline{y}_n(\xi;r)\right)(d\xi)^\alpha \\ \overline{y}_{n+1}(t;r) = \overline{y}_n(t;r) - \frac{1}{\Gamma(\alpha+1)} \int_0^t \left(\frac{d^\alpha \overline{y}_n(\xi;r)}{d\xi^\alpha} + \overline{y}_n(\xi;r)\right)(d\xi)^\alpha \end{cases}.$$

Start from

$\underline{y}_0(t;r) = (0.5 + 0.5r)$,

$\overline{y}_0(t;r) = (1.5 - 0.5r)$.

We can obtain

$\underline{y}_1(t;r) = (0.5 + 0.5r)\left(1 - \frac{t^\alpha}{\Gamma(\alpha+1)}\right)$,

$\overline{y}_1(t;r) = (1.5 - 0.5r)\left(1 - \frac{t^\alpha}{\Gamma(\alpha+1)}\right)$,

$\underline{y}_2(t;r) = (0.5 + 0.5r)\left(1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right)$,

$\overline{y}_2(t;r) = (1.5 - 0.5r)\left(1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right)$,

$\vdots$

and so on. The *n*th approximate fuzzy solution of the method approximates to the exact series solution. So, we approximate the fuzzy solutions

$$\underline{y}(t;r) = \lim_{n\to\infty} \underline{y}_n(t;r) \text{ and } \overline{y}(t;r) = \lim_{n\to\infty} \overline{y}_n(t;r).$$

In Figure 4, approximate fuzzy solution $[y(t)]^r = \left[\underline{y}_{10}(t;r), \overline{y}_{10}(t;r)\right]$ using VIM is plotted for $\alpha = 0.5$

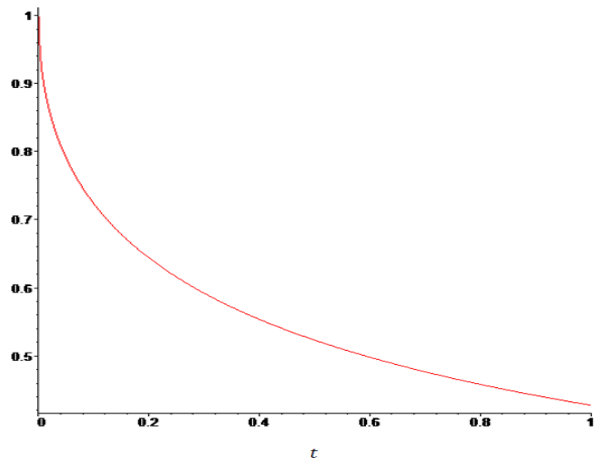

Figure 3. Exact solution

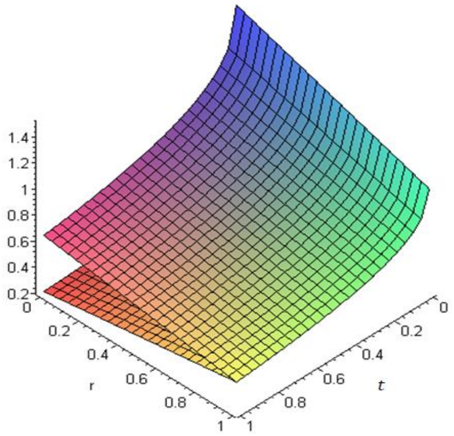

Figure 4. The approximate fuzzy solutions to the VIM

VI. CONCLUSION

In this paper, the variational iteration method was investigated to solve the fractional order fuzzy Riccati differential equations. The solution we have achieved by the variational iteration method is an infinite power series, which, can be expressed in an implicit form with appropriate fuzzy initial condition, i.e. the approximate fuzzy solution. The final numerical findings presented in tables above indicate that the variational iteration method is an efficient instrument to solving fractional order fuzzy Riccati differential equations.

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Table 1. The numerical fuzzy solution to the VIM ($\underline{y}(t;r) \cong \underline{y}_3(t;r)$)

	$\underline{y}(t;r)$											y_{Exact}
r \ t	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.4836
0.1	0.4408	0.4514	0.4615	0.4712	0.4804	0.4891	0.4974	0.5053	0.5128	0.5198	0.5265	0.6652
0.2	0.5687	0.6044	0.6375	0.6679	0.6959	0.7215	0.745	0.7663	0.7857	0.8032	0.8190	0.8591
0.3	0.5704	0.6490	0.7199	0.7835	0.8402	0.8905	0.9348	0.9736	1.0072	1.0361	1.0607	1.0586
0.4	0.4529	0.5971	0.7243	0.836	0.9332	1.017	1.0885	1.1488	1.1987	1.2393	1.2713	1.2558
0.5	0.2082	0.4455	0.6519	0.8298	0.9814	1.109	1.2148	1.3009	1.3691	1.4213	1.4593	1.4432
0.6	-0.1770	0.1864	0.4988	0.7641	0.9864	1.1698	1.3181	1.4348	1.5234	1.5871	1.6289	1.6148
0.7	-0.7200	-0.191	0.2589	0.6362	0.9480	1.2007	1.4006	1.5535	1.6647	1.7397	1.7831	1.7667
0.8	-1.4370	-0.698	-0.075	0.4419	0.8644	1.2018	1.4636	1.6587	1.7953	1.8812	1.9239	1.8970
0.9	-2.3480	-1.348	-0.512	0.1765	0.7333	1.1726	1.5077	1.7517	1.9165	2.0133	2.0527	2.0061
1.0	-3.4710	-2.155	-1.061	-0.166	0.5520	1.1119	1.5331	1.8334	2.0295	2.1371	2.1706	2.0953

Table 2. The numerical fuzzy solution to the VIM ($\overline{y}(t;r) \cong \overline{y}_3(t;r)$)

$t \backslash r$	$\overline{y}(t;r)$											y_{Exact}
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.4836
0.1	0.5754	0.5718	0.5680	0.5639	0.5595	0.5548	0.5498	0.5445	0.5389	0.5329	0.5265	0.6652
0.2	0.9009	0.8976	0.8935	0.8883	0.8822	0.8749	0.8665	0.8567	0.8456	0.8331	0.8190	0.8591
0.3	1.1338	1.1366	1.1377	1.1370	1.1342	1.1292	1.1217	1.1113	1.0979	1.0811	1.0607	1.0586
0.4	1.2870	1.3013	1.3134	1.3229	1.3293	1.3322	1.3309	1.3248	1.3134	1.2958	1.2713	1.2558
0.5	1.3664	1.3972	1.4257	1.4511	1.4726	1.4892	1.4999	1.5035	1.4989	1.4846	1.4593	1.4432
0.6	1.3762	1.4280	1.4777	1.5244	1.5666	1.6029	1.6315	1.6506	1.6581	1.6517	1.6289	1.6148
0.7	1.3203	1.3965	1.4718	1.5447	1.6131	1.6750	1.7276	1.7681	1.7933	1.7996	1.7831	1.7667
0.8	1.2020	1.3055	1.4101	1.5136	1.6134	1.7065	1.7892	1.8572	1.9059	1.9300	1.9239	1.8970
0.9	1.0246	1.1575	1.2944	1.4325	1.5686	1.6984	1.8171	1.9187	1.9969	2.0443	2.0527	2.0061
1.0	0.7911	0.9549	1.1265	1.3028	1.4796	1.6514	1.8119	1.9534	2.0671	2.1432	2.1706	2.0953

Table 3. The numerical fuzzy solution to the VIM ($\underline{y}(t;r) \cong \underline{y}_{10}(t;r)$)

$t \backslash r$	$\underline{y}(t;r)$											y_{Exact}
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	
0.0	0.5000	0.5500	0.6000	0.6500	0.7000	0.7500	0.8000	0.8500	0.9000	0.9500	1.0000	1.0000
0.1	0.3618	0.398	0.4341	0.4703	0.5065	0.5427	0.5789	0.615	0.6512	0.6874	0.7236	0.7236
0.2	0.3219	0.3541	0.3863	0.4185	0.4507	0.4828	0.5150	0.5472	0.5794	0.6116	0.6438	0.6438
0.3	0.296	0.3256	0.3552	0.3848	0.4144	0.4440	0.4736	0.5032	0.5328	0.5624	0.5920	0.5920
0.4	0.2768	0.3045	0.3322	0.3599	0.3875	0.4152	0.4429	0.4706	0.4983	0.5259	0.5536	0.5536
0.5	0.2616	0.2878	0.3139	0.3401	0.3663	0.3924	0.4186	0.4447	0.4709	0.4971	0.5232	0.5232
0.6	0.2491	0.2740	0.2989	0.3238	0.3487	0.3736	0.3985	0.4235	0.4484	0.4733	0.4982	0.4980
0.7	0.2385	0.2624	0.2862	0.3101	0.3339	0.3578	0.3817	0.4055	0.4294	0.4532	0.4771	0.4767
0.8	0.2295	0.2524	0.2754	0.2983	0.3213	0.3442	0.3672	0.3901	0.4131	0.436	0.4590	0.4582
0.9	0.2217	0.2439	0.2661	0.2882	0.3104	0.3326	0.3547	0.3769	0.3991	0.4213	0.4434	0.4420
1.0	0.2150	0.2365	0.2580	0.2795	0.3010	0.3225	0.3440	0.3655	0.3870	0.4086	0.4301	0.4276

Table 4. The numerical fuzzy solution to the VIM ($\overline{y}(t;r) \cong \overline{y}_{10}(t;r)$)

$t \backslash r$	$\overline{y}(t;r)$											y_{Exact}
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	
0.0	1.5000	1.4500	1.4000	1.3500	1.3000	1.2500	1.2000	1.1500	1.1000	1.0500	1.0000	1.0000
0.1	1.0854	1.0492	1.0130	0.9768	0.9407	0.9045	0.8683	0.8321	0.7959	0.7598	0.7236	0.7236
0.2	0.9657	0.9335	0.9013	0.8691	0.8369	0.8047	0.7725	0.7404	0.7082	0.676	0.6438	0.6438
0.3	0.8880	0.8584	0.8288	0.7992	0.7696	0.7400	0.7104	0.6808	0.6512	0.6216	0.5920	0.5920
0.4	0.8304	0.8028	0.7751	0.7474	0.7197	0.6920	0.6643	0.6367	0.6090	0.5813	0.5536	0.5536
0.5	0.7848	0.7587	0.7325	0.7063	0.6802	0.6540	0.6279	0.6017	0.5755	0.5494	0.5232	0.5232
0.6	0.7473	0.7224	0.6975	0.6725	0.6476	0.6227	0.5978	0.5729	0.5480	0.5231	0.4982	0.4980
0.7	0.7156	0.6917	0.6679	0.644	0.6202	0.5963	0.5725	0.5486	0.5248	0.5009	0.4771	0.4767
0.8	0.6885	0.6655	0.6426	0.6196	0.5967	0.5737	0.5508	0.5278	0.5049	0.4819	0.4590	0.4582
0.9	0.6651	0.6430	0.6208	0.5986	0.5765	0.5543	0.5321	0.5099	0.4878	0.4656	0.4434	0.4420
1.0	0.6451	0.6236	0.6021	0.5806	0.5591	0.5376	0.5161	0.4946	0.4731	0.4516	0.4301	0.4276