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On The Exact Quotient Of Division By Zero

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***Abstract*— This paper aims to present the solution to the most significant problem in all of analysis, namely, the problem of assigning a precise quotient for the division by zero, $a/0$. It is universally acknowledged that if a and b are two integers where $b \neq 0$, the fraction a/b, when evaluated, gives rise to only one rational quotient. But, here in analysis, at least three quotients have been assigned to the fraction $a/0$ by various departments of analysis. Moreover, so much hot debate has emerged from the discussion which has arisen from this subject. It is, therefore, the purpose of this paper to furnish the exact quotient for the special and most significant case of division by zero, the fraction $1/0$.**

***Index Terms*—Significant Problem, Analysis, Fraction, Exact Quotient, Division By Zero**

I. Introduction

A most fundamental and significant problem of mathematics for centuries wholly obscured by its complications is that of giving meaning to the division of a finite quantity by zero. It is easily seen that $a + 0 = a, a - 0 = a,$ and $a \times 0 = 0$. But, when it is required to evaluate $a \div 0$, a great difficulty arises as there is no assignable quotient.

Various noble efforts have indeed been made to assign a precise quotient for $a \div 0$ and some have proved to be stepping stones. The Indian mathematician Brahmagupta (born 598) appeared to be the first to attempt a definition for $a \div 0$. In his *Brahmasphula-siddhanta*, he spoke of $a \div 0$ as being the fraction $a/0$. Read this great mathematician: "Positive or negative numbers when divided by zero is a fraction with the zero as denominator". He seemed to have believed that $a/0$ is irreducible. In 1152, another ingenious Indian mathematician Bhaskara II improved on Brahmagupta's notion of division of a finite by zero, calling the fraction $a/0$ an infinite quantity. In his *Bijaganita* he remarked: ''A quantity divided by zero becomes a fraction the denominator of which is zero. This fraction is termed an infinite quantity''.

The illustrious English mathematician at Oxford John Wallis introduced the form $1/0 = \infty$, being the first to use the famed symbol ∞ for infinity in mathematics. In his 1655 *Arithmetica Infinitorum*, he asserted that

$$\cdots < \frac{1}{3} < \frac{1}{2} < \frac{1}{1} < \frac{1}{0} < \frac{1}{(-1)} < \cdots$$

where he considered fractions of the form $1/(-n)$ greater than the infinite quantity $1/0$. Leonhard Euler, one of the most prolific mathematicians of all times, demonstrated that ∞ and 0 are multiplicative inverses of each other. We read this genius in his excellent book *Elements of Algebra*:

The fraction $1/\infty$ represents the quotient resulting from the division of the dividend 1 by the divisor ∞. We know that if we divide 1 by the quotient $1/\infty$ which is equal to nothing, we obtain again the divisor ∞. Hence we acquire a new idea of infinity and learn that it arises from the division of 1 by 0 so that we are thence authorized in saying that 1 divided by 0 expresses a number infinitely great or ∞.

The prime goal of this paper is to complete the works of the above mentioned *connoisseurs* of division of a finite quantity by zero. Here we shall clearly show that $1/0$, which may be looked upon as the foremost of all divisions of finite quantities by zero, is equal to the actual infinite quantity $(-1)!$ which, as we shall also show, equals the infinite number $1000\ldots$.

ON THE ACTUAL INFINITE $(-1)!$

One product of numbers which occurs so frequently in applications is the factorial. For any positive integer n, the product of all positive integers from 1 up through n is called n factorial, and is denoted by the factorial function $n!$. The factorial function is so familiar and well known to all that many will regard its repetition quite superfluous. Still I regard its discussion as indispensable to prepare properly for the main question. For the way in which we define the factorial function is based directly upon only the positive integers. The factorial function, we say, is

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n. \tag{2.1}$$

The first few factorials are

a) $1! = 1, \quad 2! = 1 \cdot 2, \quad 3! = 1 \cdot 2 \cdot 3, \quad 4! = 1 \cdot 2 \cdot 3 \cdot 4.$

The result which arises from the factorials of positive integers are all positive integers; for $1! = 1, 2! = 2, 3! = 6, 4! = 24, 5! = 120$ and so on to infinity.

We now come to the chief question about the factorial: What is $(-1)!$? It will be useful to begin answering this question by considering the factorial function. Multiplying both sides of eq. (2.1) by $(n+1)$ gives

$$(n+1) \cdot n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n \cdot (n+1)$$
$$= (n+1)!.$$

Thus, we obtain the recurrence formula for the factorial:

$$n! = \frac{(n+1)!}{n+1}. \tag{2.2}$$

Letting $n = 3, 2, 1, 0,$ and -1, we get the following pattern of numbers:

$$3! = \frac{4!}{4}, \quad 2! = \frac{3!}{3}, \quad 1! = \frac{2!}{2}, \quad 0! = \frac{1!}{1}, \quad \text{and} \quad (-1)! = \frac{0!}{0}.$$

From this pattern of numbers, it is evident that $0! = 1$ and $(-1)! = 1/0$. What a picture we have here of $(-1)!$! It is the quotient which arises from the division of unity by the absolute zero.

Every artifice of ingenuity may be employed to blunt the sharp edge of this identity $(-1)! = 1/0$ and to explain away the obvious meaning of $1/0$. Here we learn at least three things. First, that $(-1)!$ is the multiplicative inverse or reciprocal of 0. Second, that the product $0 \cdot (-1)! = 1$. Third, that the infinitesimal $1/(-1)!$ equals the absolute zero 0. (How concisely do this identity dispose of the sophistries and equivocations of all who would make infinitesimals refer to only nonzero numbers less than any finite positive numbers!)

Having seen that $(-1)!$ is the quotient arising from $1/0$, we now inquire into the numerical value that will arise from the evaluation of $(-1)!$. The value of $0!$ is always taken to be, as a convention, unity. This fact, which we have proved to be true using the aforementioned recurrence relation (2.2) for the factorial, may also be obtained by numerical analysis. For if we use the computer to compute the values of the factorials of $0.1, 0.01, 0.001, \ldots$ whose limit is 0, we shall obtain the data given in the table below.

n	$n!$
0.1	$0.95135076986687318\,3629\ldots$
0.01	$0.99432585119150603\,7135\ldots$
0.001	$0.99942377248459546\,6114\ldots$
0.0001	$0.99994228832316241\,9080\ldots$
$\vdots$	$\vdots$

The figures in the second column of this table approach unity as $n \to 0$. We may conclude from this that $0! = 1$.

An understanding of this fact prepares us for the assigning of a numerical value for the infinite $(-1)!$. We can continue to use our numerical method of reasoning. The starting point is the computation of the factorials of $-0.9, -0.99, -0.999, \ldots$ whose limit is -1. The results from our computer are put in the table below.

n	$n!$
- 0.9	$10 \times 0.95135076986687318\,3629\ldots$
- 0.99	$100 \times 0.99432585119150603\,7135\ldots$
- 0.999	$1000 \times 0.99942377248459546\,6114\ldots$
- 0.9999	$10000 \times 0.99994228832316241\,9080\ldots$
?	?

The figures in the second column of this table approach $1000\ldots$ as $n \to -1$. Our new conclusion is then

$$(-1)! = 1000\ldots.$$

It should be noted that the number of zeros in $1000\ldots$ equals the number of nines in $0.999\ldots$. Had we world enough and time, we would write down all the zeros in this actual infinite $1000\ldots$.

The majority of my readers will be very much amazed in learning that by writing

$$(-1)! = 1000\ldots$$

the secret of infinity is to be revealed. To this I may say I am pleased if everybody finds the above result so obvious. It is a clear path which leads to this conclusion. We cannot show here how abundant and fruitful the consequences of this conclusion have proved. Its applications lead to simple, convincing and intuitive explanations of facts previously incoherent and misunderstood.

It is expedient that we give a glimpse of the arithmetic of infinity here that we may see the greatness of the utility of the infinite $1000\ldots$. When an integer, say n, is divided by the absolute zero, the quotient is expressed as

$$\frac{n}{0} = n \times \frac{1}{0} = n \times (-1)! = n \times 1000\ldots.$$

Setting $n = 1, 2, 3, \ldots$, we get

$$\frac{1}{0} = 1 \times 1000\ldots = 1000\ldots$$

$$\frac{2}{0} = 2 \times 1000\ldots = 2000\ldots$$

$$\frac{3}{0} = 3 \times 1000\ldots = 3000\ldots$$

and so on. It follows from these that the creation of a precise and consistent arithmetic of infinity may be possible; for it is now very clear that

$$\frac{1}{0} + \frac{2}{0} = 1000\ldots + 2000\ldots = 3000\ldots$$

$$\frac{2}{0} - \frac{3}{0} = 2000\ldots - 3000\ldots = -1000\ldots$$

$$\frac{3}{0} \times \frac{2}{0} = 3000\ldots \times 2000\ldots = 6 \times (1000\ldots)^2$$

$$\frac{1}{0} \div \frac{2}{0} = 1000\ldots \div 2000\ldots = \frac{1000\ldots}{2000\ldots} = \frac{1}{2}$$

and so on.We might give examples of all the common rules of arithmetic that pertain to finite numbers and show how they may be carried out by infinite numbers and also how they may be performed by easy operations with computers and calculators, but as this may be very elaborate we omit them in the interest of brevity.

I close this section with an interesting application of the result $1/1000\ldots = 0$ so that the reader may not entertain any doubt concerning all he has been instructed of here. It is claimed that the tangent of $90°$ is undefined or meaningless and so cannot be assigned any numerical value. But we shall show straight away that this is not the case. Suppose we wish to find $\lim_{x\to 90^\circ}\tan x$. We construct the table of values of $\tan x$ as $x \to 90^\circ$.

Table 3. Values of $\tan x$ for $x \to 90^\circ$.

x	$\tan x$
89.9°	$57.29572134 \ldots \times 10$
89.99°	$57.29577893 \ldots \times 100$
89.999°	$57.29577951 \ldots \times 1000$
$\vdots$	$\vdots$

On the basis of the information provided in the table, we say that as $x \to 90^\circ$

$$\lim_{x\to 90^\circ}\tan x = 57.295779513082 \ldots \times 1000 \ldots$$

which, with the understanding that

$$57.295779513082 \ldots = \frac{180}{\pi},$$

becomes

$$\lim_{x\to 90^\circ}\tan x = \frac{180}{\pi} \times 1000 \ldots$$

We may be filled with joy to confirm this result by taking another pathway. Familiar to us is the identity

$$\tan x = \frac{\sin x}{\cos x}$$

which, setting $x = 90^\circ$, becomes

$$\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ}.$$

To find $\tan 90^\circ$ is equivalent to finding the ratio of $\sin 90^\circ$ to $\cos 90^\circ$. It is easily seen that $\sin 90^\circ = 1$, but it will shock the reader to learn here that $\cos 90^\circ \neq 0$. We begin by constructing a table of values of $\cos x$ for $x \to 90^\circ$.

Table 4. Values of $\cos x$ for $x \to 90^\circ$.

x	$\cos x$
89.9°	$0.0174532836 \ldots /10$
89.99°	$0.0174532924 \ldots /100$
89.999°	$0.0174532925 \ldots /1000$
$\vdots$	$\vdots$

On the basis of the information provided in the table, we say that as $x \to 90^\circ$

$$\lim_{x\to 90^\circ}\cos x = 0.0174532925199 \ldots /1000 \ldots$$

which, understanding that $0.0174532925199 \ldots = \pi/180$ and $1/0 = 1000 \ldots$, becomes

$$\lim_{x\to 90^\circ}\cos x = \frac{\pi/180}{1000 \ldots} = \frac{\pi}{180}\cdot 0.$$

Thus, the value of $\tan 90^\circ$ is

$$\tan 90^\circ = \frac{1}{\frac{\pi}{180}\cdot 0} = \frac{180}{\pi \cdot 0}$$

which, setting $1/0 = 1000 \ldots$, becomes our required result

$$\tan 90^\circ = \frac{180}{\pi} \times 1000 \ldots .$$

I must apprise the reader here that the numerical value of the tangent of 90° varies with the value of the variable associated with the angle under consideration. As a way of an illustration of what we have just said, let us find the limit

$$\lim_{x\to 90^\circ}\tan\left(\frac{x^2}{90^\circ}\right).$$

We construct the following table of values of $\tan(x^2/90^\circ)$ for values of $x \to 90^\circ$.

Table 5. Values of $\tan(x^2/90^\circ)$

x	$\tan(x^2/90^\circ)$
89.9°	$28.66369780 \ldots \times 10$
89.99°	$28.64948023 \ldots \times 100$
89.999°	$28.64804890 \ldots \times 1000$
$\vdots$	$\vdots$

From the information provided in the above table, we say that as $x \to 90^\circ$

$$\lim_{x\to 90^\circ}\tan\left(\frac{x^2}{90^\circ}\right) = 28.6478897565 \ldots \times 1000 \ldots .$$

The numerical value $28.6478897565 \ldots$ is without doubt equal to $90/\pi$. Therefore, we write

$$\lim_{x\to 90^\circ}\tan\left(\frac{x^2}{90^\circ}\right) = \frac{90}{\pi} \times 1000 \ldots .$$

That the reader may be more assured of what he has been studying, I present before him the problem of finding the limit

$$\lim_{x\to 90^\circ}\frac{\tan x}{\tan\left(\frac{x^2}{90^\circ}\right)}.$$

To find the value of this limit, it is necessary to apply L' Hopital's Rule since the evaluation of this limit gives rise to the indeterminate form ∞/∞. But if we apply the infinite values already computed for the limits of both the numerator and denominator of the limit in question, we obtain

$$\lim_{x\to 90^\circ}\frac{\tan x}{\tan\left(\frac{x^2}{90^\circ}\right)} = \frac{\lim_{x\to 90^\circ}\tan x}{\lim_{x\to 90^\circ}\tan\left(\frac{x^2}{90^\circ}\right)} = \frac{57.295779513082 \ldots \times 1000 \ldots}{28.6478897565 \ldots \times 1000 \ldots} = 2$$

or

$$\lim_{x\to 90^\circ}\frac{\tan x}{\tan\left(\frac{x^2}{90^\circ}\right)}=\frac{\frac{180}{\pi}\times 1000\,...}{\frac{90}{\pi}\times 1000\,...}=2.$$

The reader acquainted with L' Hopital's Rule may check the exactness of the result above. There are many more results that may present themselves here and which it would require volumes to illustrate. But, as our plan requires great brevity, we shall be obliged to omit them.

GUARANTEEING THE TRUTH OF $\mathbf{1/0=(-1)}!$

Assigning a quotient for $1/0$ had for a long time engaged the wisdom and knowledge of mathematicians, philosophers, and theologians, and the scholarly had concluded that such a fraction is meaningless or undefined. Moreover, attempts have been made to prove that the quotient of $1/0$ is not an infinite quantity, but these attempts so clearly do violence to analysis that I will not waste time in vindicating the result $1/0=(-1)!$.

One constant with which $1/0$ is so much associated is the famed constant called Euler's constant. This constant was first introduced into mathematics by Euler in his enchanting paper entitled *De progressionibus harmonis observationes* (1734/5) [10]. There Euler defined the constant in a commendable manner as

$$\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\ln(n+1)\right),$$

and computed its arithmetical value to 6 decimal places as

$$\gamma=0.577218.$$

.

Now, the starting place of this constant goes back to a difficult problem in analysis, that of finding the exact sum of the infinite series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots.$$

This problem which was first posed by Mengoli in 1650 drilled the minds of many top mathematicians until 1734 when Euler showed that the sum of the series is $\pi^2/6$. It was while he was attempting to assign a sum to the famous harmonic series

$$\sum_{k=1}^{\infty}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\cdots$$

that he discovered his constant and denoted it with the letter C, stating that it was ' worthy of serious consideration' [17], [34].

Let us now use $1/0=(-1)!$ in the derivation of the definition of Euler's constant in order to guarantee that the result $1/0=(-1)!$ is true. We begin with the familiar relation [36]

$$\int_0^m H_x\,dx=m\gamma+\ln(m!)$$

where H_x is the xth harmonic number. Noting that [36]

$$H_x=\sum_{k=1}^{\infty}\frac{x}{k(x+k)}$$

we write

$$\int_0^m\sum_{k=1}^{\infty}\frac{x}{k(x+k)}dx=m\gamma+\ln(m!)$$

which, resolving $x/k(x+k)$ into partial fractions, becomes

$$\int_0^m\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{x+k}\right)dx=m\gamma+\ln(m!)$$

which becomes

$$\sum_{k=1}^{\infty}\int_0^m\left(\frac{1}{k}-\frac{1}{x+k}\right)dx=m\gamma+\ln(m!).$$

This, evaluating $\int_0^m\left(\frac{1}{k}-\frac{1}{x+k}\right)dx$, simplifies into

$$\sum_{k=1}^{\infty}\left(\frac{m}{k}-\ln\left(\frac{m+k}{k}\right)\right)=m\gamma+\ln(m!).$$

Setting $m=-1$, we obtain

$$\sum_{k=1}^{\infty}\left(\frac{-1}{k}-\ln\left(\frac{-1+k}{k}\right)\right)=-\gamma+\ln(-1)!$$

which becomes

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}+\ln\left(\frac{k-1}{k}\right)\right)=\gamma-\ln(-1)!.$$

This result may be expressed as

$$\sum_{k=1}^{\infty}\frac{1}{k}+\sum_{k=1}^{\infty}\ln\left(\frac{k-1}{k}\right)=\gamma-\ln(-1)!$$

which gives us

$$\sum_{k=1}^{\infty}\frac{1}{k}+\ln\left(\frac{1-1}{1}\right)+\sum_{k=2}^{\infty}\ln\left(\frac{k-1}{k}\right)=\gamma-\ln(-1)!$$

which in its turn gives

$$\sum_{k=1}^{\infty}\frac{1}{k}+\ln\left(\frac{0}{1}\right)+\sum_{k=2}^{\infty}\ln\left(\frac{k-1}{k}\right)=\gamma-\ln(-1)!$$

which, setting $\ln\left(\frac{0}{1}\right)=-\ln\left(\frac{1}{0}\right)$, furnishes

$$\sum_{k=1}^{\infty}\frac{1}{k}-\ln\left(\frac{1}{0}\right)+\sum_{k=2}^{\infty}\ln\left(\frac{k-1}{k}\right)=\gamma-\ln(-1)!.$$

This result, applying our inspirational identity $1/0=(-1)!$, is equivalent to

$$\sum_{k=1}^{\infty}\frac{1}{k}-\ln(-1)!+\sum_{k=2}^{\infty}\ln\left(\frac{k-1}{k}\right)=\gamma-\ln(-1)!$$

which ultimately becomes

$$\sum_{k=1}^{\infty}\frac{1}{k}+\sum_{k=2}^{\infty}\ln\left(\frac{k-1}{k}\right)=\gamma.$$

Now the sum

$$\sum_{k=2}^{\infty}\ln\left(\frac{k-1}{k}\right)=\lim_{n\to\infty}\left(\ln\left(\frac{1}{2}\right)+\ln\left(\frac{2}{3}\right)+\ln\left(\frac{3}{4}\right)+\cdots+\ln\left(\frac{n-2}{n-1}\right)+\ln\left(\frac{n-1}{n}\right)\right)$$

$$=\lim_{n\to\infty}\left(\ln\left(\frac{1}{n}\right)\right)$$

$$=-\lim_{n\to\infty}(\ln n).$$

Similarly, the sum

$$\sum_{k=1}^{\infty}\frac{1}{k}=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right).$$

Taking these as essential steps, we obtain

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\ln(n)\right)=\gamma$$

which is Euler's original definition of γ.

Let us now give a splendid illustration of the way in which the identity $1/0=(-1)!$ may be used in analysis. Our aim at this point is to demonstrate that $\ln(-1)!$ is the sum of the harmonic series. The possibility of such a result is suggested by inspecting the Taylor series expansion

$$\ln\left(\frac{1}{1-x}\right)=x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots,$$

and letting $x=1$. Accomplishing these, we obtain the following:

$$\ln\left(\frac{1}{1-1}\right)=1+\frac{1^2}{2}+\frac{1^3}{3}+\cdots,$$

$$\ln\left(\frac{1}{0}\right)=1+\frac{1}{2}+\frac{1}{3}+\cdots.$$

Employing the identity $1/0=(-1)!$, we arrived at the required result

$$\ln(-1)!=1+\frac{1}{2}+\frac{1}{3}+\cdots=\sum_{k=1}^{\infty}\frac{1}{k}.$$

Many other proofs might be given to show that $\ln(-1)!$ is actually the sum of the harmonic series, but this is so explicit that we have thought proper not to enlarge because we cannot possibly do justice to the great subject involved.

Let us now turn to the derivation of a formula in analysis in order to give the reader an idea of the flavor of

$$\ln(-1)!=\sum_{k=1}^{\infty}\frac{1}{k}.$$

There is a very interesting formula discovered by Euler in his 1776 paper [15], which presents a beautiful means of computing γ. This formula, which reappeared in several subsequent works by many mathematicians of eminence such as Glaisher [15], Johnson [18], Bromwich [5], Srivastava [29], Lagarias [20], and Barnes and Kaufman [3], is

$$1-\gamma=\sum_{k=2}^{\infty}\frac{(\zeta(k)-1)}{k}.$$

We now proceed to derive this formula which has fascinated the industry of such a great number of mathematicians and we begin with the familiar Maclaurin series expansion of the natural logarithm of $x!$

$$\ln(x)!=-\gamma x+\sum_{k=2}^{\infty}(-1)^k\frac{\zeta(k)}{k}x^k,\qquad |x|<1.$$

We shall here violate the proviso that $|x|<1$; for if we let $x=-1$ so that $|x|=1$, an encroachment of the stipulation $|x|<1$, then we obtain the result

$$\ln(-1)!=\gamma+\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}.$$

Setting $\ln(-1)!=\sum_{k=1}^{\infty}\frac{1}{k}$, we obtain

$$\sum_{k=1}^{\infty}\frac{1}{k}=\gamma+\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}$$

which results in

$$1+\sum_{k=2}^{\infty}\frac{1}{k}-\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}=\gamma$$

which in turn furnishes our required formula

$$1-\gamma=\sum_{k=2}^{\infty}\frac{(\zeta(k)-1)}{k}.$$

To be more fully convinced of the fact that $\ln(-1)!$ is the sum of the harmonic series, we employ it again in the derivation of this same formula by taking another lane. We begin with the familiar identity [36]

$$H_x=\sum_{k=1}^{\infty}(-1)^{k+1}\,x^k\zeta(k+1)$$

and integrate both sides of it with respect to x, that is, we find

$$\int_0^m H_x\,dx=\int_0^m\sum_{k=1}^{\infty}(-1)^{k+1}\,x^k\zeta(k+1)\,dx$$

where H_x is the x th harmonic number. We apply the aforementioned familiar relation [36]

$$\int_0^m H_x\,dx=m\gamma+\ln(m!)$$

and get

$$m\gamma+\ln(m!)=\sum_{k=1}^{\infty}(-1)^{k+1}\zeta(k+1)\int_0^m x^k\,dx$$

which becomes

$$m\gamma+\ln(m!)=\sum_{k=1}^{\infty}(-1)^{k+1}\zeta(k+1)\frac{m^{k+1}}{k+1}.$$

Let us now set $m=-1$. We obtain

$$-\gamma+\ln(-1)!=\sum_{k=1}^{\infty}\frac{\zeta(k+1)}{k+1}$$

which furnishes

$$\gamma=\ln(-1)!-\sum_{k=1}^{\infty}\frac{\zeta(k+1)}{k+1}.$$

We set $k=k-1$ and get

$$\gamma=\ln(-1)!-\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}.$$

Finally, setting $\ln(-1)!=\sum_{k=1}^{\infty}\frac{1}{k}$, we obtain

$$\gamma=\sum_{k=1}^{\infty}\frac{1}{k}-\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}$$

which, taking an easily construed step, becomes our proposed formula:

$$1-\gamma=\sum_{k=2}^{\infty}\frac{(\zeta(k)-1)}{k}.$$

Therefore, it remains for us to remove any doubt which may be entertained concerning the utility of the logarithmic infinity $\ln(-1)!$, for this number being infinite, it would not be surprising if anyone should think it entirely meaningless and useless. This however is not the case. The computation involving the logarithmic infinity is of the greatest importance. When the ubiquitous harmonic series appears in any calculation or formula, we are certain that its sum is the logarithmic infinity $\ln(-1)!$.

It may not be amiss to show in this work whether or not γ is irrational. To prove or disprove the irrationality of γ has acquired extraordinary celebrity from the fact that no correct proof has been given, but there is no reason to doubt that it is possible. We shall, therefore, pursue here the proof of the irrationality of γ. We begin with the mystery of γ in which Euler has beautifully mingled the harmonic series with the natural logarithm, that is the excellent relation

$$\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\ln(n+1)\right).\quad(3.1)$$

In the language of the Nonstandard Analysis invented by the grand American logician Abraham Robinson of Yale University, let ω be the infinite positive integer for which

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{\omega}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=\ln(-1)!.$$

We rewrite (3.1) as

$$\gamma=\lim_{n\to\omega}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\ln(n+1)\right)$$

or

$$\gamma=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{\omega}-\ln(\omega+1)$$

which becomes

$$\gamma=\ln(-1)!-\ln(\omega+1)$$

which simplifies to

$$\gamma=\ln\left(\frac{(-1)!}{\omega+1}\right)$$

which, in its own turn, after finding the natural exponential of both sides, furnishes the nice result

$$e^{\gamma} = \frac{(-1)!}{\omega+1}.$$

Rearranging this as

$$\frac{e^{\gamma}}{(-1)!} = \frac{1}{\omega+1},$$

and noting that $1/(-1)! = 0,$ we obtain

$$e^{\gamma} \cdot 0 = \frac{1}{\omega+1}.$$

Now, by the renowned transfer principle, $\frac{1}{\omega+1}$ is a rational number as it is the ratio of two integers, the finite integer 1 and the infinite integer $\omega+1$. Therefore, it follows that $e^{\gamma} \cdot 0$ to which $\frac{1}{\omega+1}$ is equal is rational. Since the integer 0 is rational, it is evident that, for $e^{\gamma} \cdot 0$ to be rational, e^{γ} must be rational.

In the excellent book *An Introduction to the Theory of Numbers* [17] the Great Britain's professional mathematicians, G. H. Hardy and E. M. Wright, show that e^{x} is irrational for every rational $x \neq 0$, a result first reached by Lambert [16], [27]. To cite the proof is too great a work for us. We will, however, cite the words of one of the most eminent mathematics historians, F. Cajori:

In 1761 Lambert communicated to the Berlin Academy a memoir (published 1768), in which he proves rigorously that π is irrational. It is given in simplified form in Note IV of A. M., Legendre's Geometric, where the proof is extended to π^2. Lambert proved that if x is rational, but not zero, then neither e^{x} nor $\tan x$ can be a rational number; since $\tan(\pi/4) = 1$, it follows that $\pi/4$ or π cannot be rational.

If, therefore, $\gamma \neq 0$ were rational, then e^{γ} would be irrational, a contradiction, since e^{γ} as we have seen, is rational. Thus γ is an irrational number, incapable of being written as a ratio of two integers.

Let us inquire into the value of $\omega+1$. If we re-express

$$e^{\gamma} = \frac{(-1)!}{\omega+1}$$

as

$$\omega+1 = e^{-\gamma}(-1)!$$

and noting that $e^{-\gamma} = 0.561459483566885169824143 2147\ldots$ and $(-1)! = 1000\ldots$ as it was pointed out in Section 2, we have

$$\omega+1 = 0.5614594835668851698241432147\ldots \times 1000\ldots$$

$$= 5614594835668851698241432147\ldots.$$

Thus the numerical value of $\omega+1$ is $5614594835668851698241432147\ldots$ an infinite integer less than $1000\ldots$. The number of digits in the number $5614594835668851698241432147\ldots$ equals the number of zeros in $1000\ldots$.

We inquire whether the use of the word "undefined" for the expression $1/0$ is proper. We have already agreed that $\omega+1$ is an actual infinite number. Therefore, no one will have any difficulty in comprehending that $2(\omega+1)$, $3(\omega+1)$, $4(\omega+1),\ldots$ are also infinite numbers. Moreover, it is very clear that $e^{\gamma}(\omega+1)$ is an infinite number between $\omega+1$ and $2(\omega+1)$ since e^{γ} is a real number $1.7810724179901979852365041031071795491696452143034 3\ldots$ between 1 and 2. If we admit that $e^{\gamma}(\omega+1)$ is actually a number, though infinite, I do not see how $1/0$ may be meaningless or undefined. For if we begin again with

$$e^{\gamma} = \frac{(-1)!}{\omega+1},$$

rewrite it as

$$e^{\gamma}(\omega+1) = (-1)!$$

and set $(-1)! = 1/0$, we obtain the shocking result

$$e^{\gamma}(\omega+1) = \frac{1}{0}.$$

But we have said before that $e^{\gamma}(\omega+1)$ is an infinite number. Therefore, $1/0$ which the mathematical community has hitherto termed undefined is actually a number and is infinite. What a glorious subject is now presented to our view! But we must leave it, for our limits remind us that we must be brief.

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