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Complex of Lascoux in Partition (6,6,3)

Haytham Razooki Hassan, Mays Majid Mohammed

***Abstract*— In this paper, the complex of Lascoux in the case of partition (6,6,3) has been studied by using diagrams, divided power of the place polarization $\partial_{ij}^{(k)}$,Capelli identities and the idea of mapping cone.**

***Index Terms*— Divided power algebra, Resolution of Weyl module, Place polarization, Mapping Cone**

I. Introduction

Let R be the commutative ring with 1, F be a free module and D_sF be the divided power of degree s. Another type of maps are used in Buchsbaum whose images define schur and Weyl modules which send an element $a \otimes b$ of $D_{p+k} \otimes D_{q-k}$ to $\sum a_p \otimes a'_k b$, where $\sum a_p \otimes a'_k$ is the component of the diagonal of a in $D_p \otimes D_k$,the generalization of this map to ones , where there more factors were called in the 'box map'.

The complex of characteristic zero is studied in [3],[4] and [5] in the partition (2,2,2) ,(3,3,3) and (4,4,3), using this modified and the letter place methods [3] , In this paper we study the complex of Lasoux in the case of partition (6,6,3) as a diagram by using the idea of the mapping Cone [6] , and the map $\partial_{ij}^{(k)}$ which means the k^{th} divided power of the place polarization ∂_{ij} where *j* must be less than *I* with it's Caplli identities [1] , specificly in this work we used only the following identities

$$\partial_{32}^{(l)}\partial_{21}^{(k)} = \sum\nolimits_{\alpha\geq 0}\partial_{21}^{(k-\alpha)}\partial_{32}^{(l-\alpha)}\partial_{31}^{(\alpha)} \qquad (1.1)$$

$$\partial_{21}^{(k)}\partial_{32}^{(l)} = \sum\nolimits_{\alpha\geq 0}(-1)^{\alpha}\partial_{32}^{(l-k)}\partial_{21}^{(k-\alpha)}\partial_{31}^{(\alpha)}$$

(1.2)

$$\partial_{21}^{(1)} \circ \partial_{31}^{(1)} = \partial_{31}^{(1)} \circ \partial_{21}^{(1)}$$

and

$$\partial_{32}^{(1)} \circ \partial_{31}^{(1)} = \partial_{31}^{(1)} \circ \partial_{32}^{(1)} \qquad (1.3)$$

Where ∂_{ij} is the place polarization from place *j* to place *i*.

II. The terms of Lascoux complex in the case of partition (6,6,3)

The terms of the lascoux complex are obtained from the determinantal expansion of the Jocobi-trudi matrix of the

partition. The positions of the terms of the complex are determined by the length of the permutation to which they

Haytham R. Hassan, PhD. in mathematics from roma university. Asst.prof . Department of Math science , Al-Mustansiriyah University / Science College , Baghdad, Iraq , Mobile No.009647711491460

Mays M. Mohammed, Master student,Department of Math science, Al-Mustansiriyah University / Science College , Baghdad, Iraq ,Mobile NO. 009647714187543.

correspond [2],[3]. Now in the 6,6,3), we have the following matrix: ꝛ=(case of the partition

$$\begin{bmatrix} D_6 & D_5 & D_1 \\ D_7 & D_6 & D_2 \\ D_8 & D_7 & D_3 \end{bmatrix}$$

Then the Lascoux complex has the correspondence between it's terms as follows:

$D_6F \otimes D_6F \otimes D_3F \leftrightarrow identity$
$D_5F \otimes D_7F \otimes D_3F \leftrightarrow (12)$
$D_6F \otimes D_2F \otimes D_7F \leftrightarrow (23)$
$D_5F \otimes D_2F \otimes D_8F \leftrightarrow (123)$
$D_1F \otimes D_7F \otimes D_7F \leftrightarrow (132)$

So, the complex of Lascoux in the case of the partition $\lambda = (6,6,3)$ has the form:-

$$\begin{array}{ccccc} & D_8F \otimes D_5F \otimes D_2F & & D_7F \otimes D_5F \otimes D_3F & \\ D_8F \otimes D_6F \otimes D_1F \rightarrow & \oplus & \rightarrow & \oplus & \rightarrow D_6F \otimes D_6F \otimes D_3F \\ & D_7F \otimes D_7F \otimes D_1F & & D_6F \otimes D_7F \otimes D_2F & \end{array}$$

III. The complex of Lascoux as a diagram

Consider the following diagram :

$$\begin{tikzcd} D_8F \otimes D_6F \otimes D_1F \arrow[r, "S_1"] \arrow[d, "b_1"'] \arrow[dr, phantom, "\mathrm{M}"] & D_8F \otimes D_5F \otimes D_2F \arrow[r, "S_2"] \arrow[d, "b_2"'] \arrow[dr, phantom, "\mathrm{N}"] & D_7F \otimes D_5F \otimes D_3 \arrow[d, "b_3"'] \\ D_7F \otimes D_7F \otimes D_1F \arrow[r, "t_1"] & D_6F \otimes D_7F \otimes D_2F \arrow[r, "t_2"] & D_6F \otimes D_6F \otimes D_3F \end{tikzcd}$$

So, if we define

$S_1 : D_8F \otimes D_6F \otimes D_1F \longrightarrow D_8F \otimes D_5F \otimes D_2F$
as, $S_1(V) = \partial_{32}(V)$ $\quad$ $where ; V \in D_8F \otimes D_6F \otimes D_1F$
$b_1 : D_8F \otimes D_6F \otimes D_1F \longrightarrow D_7F \otimes D_7F \otimes D_1F$
as, $b_1(V) = \partial_{21}(V)$ $\quad$ $where\ V \in D_8F \otimes D_6F \otimes D_1F$
$b_2 : D_8F \otimes D_5F \otimes D_2F \longrightarrow D_6F \otimes D_7F \otimes D_2F$
as, $b_2(V) = \partial_{21}^{(2)}(V)$ $\quad$ $where\ V \in D_8F \otimes D_5F \otimes D_2F$

Now , we have to define the following map which makes the diagram M commutative:

$t_1 : D_7F \otimes D_7F \otimes D_1F \longrightarrow D_6F \otimes D_7F \otimes D_2F$

So we have:

$t_1 \circ b_1 = b_2 \circ S_1$

Which implies that

$t_1 \circ \partial_{21} = \partial_{21}^{(2)} \circ \partial_{32}$

Now we use Capelli identities from

$$\partial_{21}^{(2)} \circ \partial_{32} = \partial_{32} \circ \partial_{21}^{(2)} - \partial_{31} \circ \partial_{21}$$
$$= \left(\tfrac{1}{2}\partial_{32} \circ \partial_{21} - \partial_{31}\right) \circ \partial_{21}$$

Thus , $t_1 = \frac{1}{2}\partial_{32} \circ \partial_{21} - \partial_{31}$

On the other hand, if we define

$t_2 : D_6F \otimes D_7F \otimes D_2F \longrightarrow D_6F \otimes D_6F \otimes D_3F$
$t_2(v) = \partial_{32}(v)$ $\quad$ where; $v \in D_6F \otimes D_7F \otimes D_2F$
and $b_3 : D_7F \otimes D_5F \otimes D_3F \longrightarrow D_6F \otimes D_6F \otimes D_3F$
$b_3(v) = \partial_{21}(v)$ $\quad$ *where;* $v \in D_7F \otimes D_5F \otimes D_3F$ $\quad$ *as,*

Now we need to define S_2 to make the diagram N commute:

$S_2 : D_8F \otimes D_5F \otimes D_2F \longrightarrow D_7F \otimes D_5F \otimes D_3F$

Such that $b_3 \circ S_2 = b_3 \circ S_2$ *i.e.* $\partial_{21} \circ S_2 = \partial_{32} \circ \partial_{21}^{(2)}$

Again by using Caplli identities we get

$$\partial_{32} \circ \partial_{21}^{(2)} = \partial_{21}^{(2)} \circ \partial_{32} + \partial_{21} \circ \partial_{31}$$
$$= \partial_{21}\left(\tfrac{1}{2}\partial_{21} \circ \partial_{32} + \partial_{31}\right)$$

Then $S_2 = \frac{1}{2}\partial_{21} \circ \partial_{32} + \partial_{31}$

Now consider the following diagram :

$$\begin{tikzcd}
D_8F \otimes D_6F \otimes D_1F \arrow[r, "s_1"] \arrow[d, "b_1"'] & D_8F \otimes D_5F \otimes D_2F \arrow[r, "S_2"] \arrow[d, phantom, "H"'] \arrow[dr, phantom, "G"] & D_7F \otimes D_5F \otimes D_3F \arrow[d, "b_3"] \\
D_7F \otimes D_7F \otimes D_1F \arrow[r, "t_1"] \arrow[urr, "z"] & D_6F \otimes D_7F \otimes D_2F \arrow[r, "t_2"] & D_6F \otimes D_6F \otimes D_3F
\end{tikzcd}$$

Define $z : D_7F \otimes D_7F \otimes D_1F \longrightarrow D_7F \otimes D_5F \otimes D_3F$

By $z(v) = \partial_{32}^{(2)}$ where $v \in D_7F \otimes D_7F \otimes D_1F$.

Proposition 3.1:- The diagram H is commutative.

Proof :- To prove H is commutative , we need to prove $S_2 \circ S_1 = z \circ b_1$

$$S_2 \circ S_1 = \left(\frac{1}{2}\partial_{21} \circ \partial_{32} + \partial_{31}\right) \circ \partial_{32}$$
$$= \partial_{21} \circ \partial_{32}^{(2)} + \partial_{32} \circ \partial_{31}$$
$$= \partial_{32}^{(2)} \circ \partial_{21} - \partial_{32} \circ \partial_{31} + \partial_{32} \circ \partial_{31}$$
$$= \partial_{32}^{(2)} \circ \partial_{21}$$
$$= z \circ \partial_{21} .$$

■ **Proposition 3.2:-** The diagram G is commutative

Proof :-

$$t_2 \circ t_1 = \left(\frac{1}{2}\partial_{21} \circ \partial_{32} + \partial_{31}\right) \circ \partial_{32}$$
$$= \partial_{21}\partial_{32}^{(2)} + \partial_{32}\partial_{31}$$
$$= \partial_{32}^{(2)} \circ \partial_{21} - \partial_{32} \circ \partial_{31} + \partial_{32} \circ \partial_{31}$$
$$= \partial_{32}^{(2)} \circ \partial_{21}$$
$$= z \circ \partial_{21} .$$

■

Finally by using the mapping Cone we can define the maps σ_1, σ_2 and σ_3 where:

$$\sigma_3 : D_8F \otimes D_6F \otimes D_1F \longrightarrow \begin{array}{c} D_8F \otimes D_5F \otimes D_2F \\ \oplus \\ D_7F \otimes D_7F \otimes D_1F \end{array}$$

$$\sigma_2 : \begin{array}{c} D_7F \otimes D_5F \otimes D_3F \\ \oplus \\ D_7F \otimes D_7F \otimes D_1F \end{array} \longrightarrow \begin{array}{c} D_8F \otimes D_5F \otimes D_2F \\ \oplus \\ D_6F \otimes D_7F \otimes D_2F \end{array}$$

and

$$\sigma_1 : \begin{array}{c} D_7F \otimes D_5F \otimes D_3F \\ \oplus \\ D_6F \otimes D_7F \otimes D_2F \end{array} \longrightarrow D_6F \otimes D_6F \otimes D_3F$$

$D_8F \otimes D_5F \otimes D_2F$

$D_7F \otimes D_5F \otimes D_3F$

$0 \longrightarrow D_8F \otimes D_6F \otimes D_1F \longrightarrow$ $\oplus$ $\longrightarrow$
$\oplus$ $\longrightarrow D_6F \otimes D_6F \otimes D_3F$
$D_6F \otimes D_7F \otimes D_2F$
$D_7F \otimes D_7F \otimes D_1F$

by

$\bullet \sigma_3(x) = (s_1(x), b_1(x))$; $\forall x \in D_8F \otimes D_6F \otimes D_1F$

$D_8F \otimes D_5F \otimes D_2F$

$\bullet$ $\sigma_2((x_1, x_2)) = (s_2(x_1) - z(x_2), b_1(x_2) - b_2(x_1))$; $\forall (x_1, x_2) \in$ $\oplus$

$D_7F \otimes D_7F \otimes D_1F$

$D_7F \otimes D_5F \otimes D_3F$

$\bullet \sigma_1((x_1, x_2)) = (b_3(x_1) + t_2(x_2))$; $\forall (x_1, x_2) \in$ $\oplus$

$D_6F \otimes D_7F \otimes D_2F$

Propsition 3.3:

$D_8F \otimes D_5F \otimes D_2$ F

$D_7F \otimes D_5F \otimes D_3F$

0

$\rightarrow$

$D_8F \otimes D_6F \otimes D_1F \xrightarrow{\sigma_3}$ $\oplus$
$\xrightarrow{\sigma_2}$ $\oplus$ $\xrightarrow{\sigma_1} D_6F \otimes D_6F \otimes D_3F$

$D_7F \otimes D_7F \otimes D_1F$ $D_6F \otimes D_7F \otimes D_2F$

is complex.

Proof:-

Since $\partial_{32}^{(1)}$ and $\partial_{21}^{(1)}$ are injective from it is definition (see [1]), then we get σ_3 is injective.

Now

$$\sigma_2 \circ \sigma_3 = \sigma_2(s_1(x), b_1(x))$$
$$= \sigma_2(\partial_{32}(x), \partial_{21}(x))$$

$$= \left(s_2(\partial_{32}(x)) - z(\partial_{21}(x)), t_1(\partial_{21}(x)) - b_2(\partial_{32}(x))\right).$$

Now

$$s_2(\partial_{32}(x)) - z(\partial_{21}(x)) = \left(\frac{1}{2}\partial_{21} \circ \partial_{32} + \partial_{31}\right) \circ \partial_{32}(x) - \partial_{32}^{(2)} \circ \partial_{21}(x)$$

$$= (\partial_{21} \circ \partial_{32}^{(2)} + \partial_{31} \circ \partial_{32} - \partial_{32}^{(2)} \circ \partial_{21})(x)$$

$$= (\partial_{21} \circ \partial_{32}^{(2)} + \partial_{32} \circ \partial_{31} - \partial_{21} \circ \partial_{32}^{(2)} - \partial_{32} \circ \partial_{31})(x)$$
$$= 0.$$

$$t_1(\partial_{21}(x)) - b_2(\partial_{32}(x)) = \left(\frac{1}{2}\partial_{32} \circ \partial_{21} - \partial_{31}\right) \circ \partial_{21}(x) - \partial_{21}^{(2)} \circ \partial_{32}(x)$$

$$= (\partial_{32} \circ \partial_{21}^{(2)} - \partial_{21} \circ \partial_{31} - \partial_{32} \circ \partial_{21}^{(2)} + \partial_{21} \circ \partial_{31})(x)$$
$$= 0.$$

So we get $(\sigma_2 \circ \sigma_3)(x) = 0.$

and

$(\sigma_1 \circ \sigma_2)(x_1, x_2) = \sigma_1(s_2(x_1) - z(x_2), t_1(x_2) - b_2(x_1))$

$= \sigma_1\left(\left(\frac{1}{2}\partial_{21} \circ \partial_{32} + \partial_{31}\right)(x_1) - \partial_{32}^{(2)}(x_2), \left(\frac{1}{2}\partial_{32} \circ \partial_{21} -\right.\right.$
$\left.\left.\partial_{31}\right)(x_2) - \partial_{21}^{(2)}(x_1)\right)$
$= \partial_{21}\left(\frac{1}{2}\partial_{21}\partial_{32} + \partial_{31}\right)(x_1) - \partial_{21} \circ \partial_{32}^{(2)}(x_2)$
$+\partial_{32}\left(\frac{1}{2}\partial_{32} \circ \partial_{21} - \partial_{31}\right)(x_2) - \partial_{32} \circ \partial_{21}^{(2)}(x_1)$
$= \left(\partial_{21}^{(2)} \circ \partial_{32} + \partial_{21} \circ \partial_{31} - \partial_{32} \circ \partial_{21}^{(2)}\right)(x_1)$
$+\left(\partial_{32}^{(2)} \circ \partial_{21} - \partial_{32} \circ \partial_{31} - \partial_{21} \circ \partial_{32}^{(2)}\right)(x_2).$
then

$(\sigma_1 \circ \sigma_2)(x_1, x_2) = (\partial_{32} \circ \partial_{21}^{(2)} - \partial_{21} \circ \partial_{31} + \partial_{21} \circ \partial_{31} - \partial_{32} \circ \partial_{21}^{(2)})(x_1)$

$+(\partial_{21} \circ \partial_{32}^{(2)} + \partial_{32} \circ \partial_{31} - \partial_{32} \circ \partial_{31} - \partial_{21} \circ \partial_{32}^{(2)})(x_2)=0.$

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