Effect of Radiative Heat Transfer on Cosmic-Ray Transport in a Rotating Cloudy Interstellar Medium

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Abstract— Radiative heat transfer on cosmic rays (cr) in a rotating cloudy interstellar medium is modelled by imposing a time dependent perturbation on the cosmic transport in the interstellar medium containing randomly distributed giant diffused molecular clouds the temperature involved are assumed to be large so that radiative heat transfer is significant. This renders the problem very nonlinear even on the assumptions of a differential approximation for the radiative flux in an adiabatic concentration and zero activated energy. When the perturbation is small, the transient flow is tackled by the laplace transform technique, giving solution, for steady state spectrum of cosmic ray in the galaxy temperature up and down stream concentration, also velocity analytical and numerical solutions were obtained for temperature. Velocity of the down and upstream concentration in cases where radiative heat transfer through the rotating medium are taken into account. The incorporation of radiative heat transfer is of particular relevance to cosmic ray transport whose temperature is usually very high which radiate a lot of heat through the media which surround them.

Index Terms— Cosmic Ray, Radiative Heat, laplace transform, interstellar medium.

I. INTRODUCTION

Modeling of cosmic rays(cr)transport in the cloudy interstellar medium containing randomly distributed giant molecular clouds have been considered in the works of (1) cowisk and wilson(1973) dogiel et al (1987)monfill et al (1985),(2)osborne et al (1987) and ptuskin et al(1990).the latter two papers the theory of diffusion in the cloudy medium which takes into account the cloud’s finite transparency for diffusion particles. Due to the high temperature involved its application in astrophysics, meteorology cannot be overemphasised .In all these investigations the problem of radiative heat transfer has been ignored unfortunately high temperature phenomena abound in cosmic ray transport which radiate a lot of heat through the media which surround them. A primary difficulty in thermal radiative heat transfer studies stems from the fact that the radiative flux is governed by an integral expression and one has to handle a nonlinear integro-differential equation. However, under fairly broad realistic assumptions the integral expression may be replaced by a differential approximation for radiation. Thus in one space coordinates \( z \), the radiative flux \( \tilde{Q} \) satisfies the nonlinear differential equation cheng (1964) similarly arrienus will be linear zed.

\[
\frac{\partial^2 \tilde{Q}'}{\partial Z'^2} - 3\alpha^2 \tilde{Q}' - 16\alpha \sigma T^3 \frac{\partial T}{\partial Z'} = 0 \quad [1]\]

\[
K_T = \alpha e^{\frac{E}{RT}} \quad [1]b
\]

Where \( T \) is the temperature of the CR \( \sigma \) is the Stefan-Boltzmann constant and \( \alpha \) is the absorption coefficient which will be assumed constant in the modelling is the chemical reaction. Taking into account the cloud’s finite transparency for diffusing particles \( \alpha \) and equation1.1a and 1.1b is approximated by

\[
\frac{\partial Q'}{\partial Z'} = 4\alpha \sigma (T^4 - T^4) \quad [2]a
\]

\[
K_r = \frac{K}{RT} (T - T) \quad [2]b
\]

in which subscript \( \infty \) will be used denote condition in the undisturbed cloudy interstellar medium. In the subsequent analysis, the mathematical formulation of the problem for finite transparent medium is presented in section 2 for the case in which the temperature of the medium suffers time \( t' \) perturbation when this perturbation is small, and the non-linear system tackled by asymptotic approximation so that the first order transient problem is solved by the laplace transform technique the results are obtained in section 3.In section 4 a discussion of the results of the previous section given.

II. NOMENCLATURE

- \( q_c \): radiative heat flux
- \( q \): complex velocity
- \( k_p \): Boltzmann constant
- \( \sigma = \text{Stefan-Boltzmann constant.} \)
- \( H_{\alpha}^2 \): constant transverse magnetic field
- \( K_{q_2} \): thermo diffusion constant
- \( k_r \): constant associated with chemical reaction in the Arrehnius term
- \( P_r \): Prandtli number
- \( R \): Radiation parameter
- \( D_j \): Diffusion parameter
- \( i = \sqrt{-1} \): complex number
- \( E \): rotation parameter
- \( M \): magnetic parameter
- \( G_{f} \): free convection parameter due to temperature
- \( D_m \): mass diffusivity
\[ S_e = \text{Soret parameter.} \]
\[ S_{sc} = \text{Schmidt number} \]
\[ (u', v') = \text{dimensional velocity component} \]
\[ (X', Y', Z') = \text{dimensional Cartesian co-ordinate} \]
\[ k = \text{thermal conductivity} \]
\[ g = \text{gravitational acceleration} \]
\[ c_p = \text{specific heat at constant pressure} \]
\[ T = \text{dimensional temperature} \]
\[ c = \text{dimensional concentration} \]
\[ T_e = \text{reservoir temperature} \]
\[ c_e = \text{reservoir concentration} \]
\[ T_w = \text{constant plate temperature} \]
\[ c_w = \text{constant plate concentration} \]
\[ T_m = \text{mean temperature} \]
\[ G_c = \text{free convection parameter due to concentration.} \]
\[ \mu = \text{the permeability} \]
\[ \alpha = \text{absorption coefficient} \]
\[ \beta = \text{coefficient of volume expansion due temperature.} \]
\[ \sigma_c = \text{electrical conductivity} \]
\[ \gamma = \text{coefficient of volume expansion due concentration.} \]
\[ \varepsilon = \text{small parameter} \]
\[ \nu = \text{kinematic viscosity.} \]
\[ \Omega = \text{angular velocity} \]
\[ \rho_{\infty} = \text{reservoir density.} \]

### III. MATHEMATICAL FORMULATIONS

We consider the cosmic ray transport (CR) in the interstellar medium containing randomly distributed giant molecular clouds with characteristic velocity \(U_0\) and rotates about the X-axis with angular velocity \(\Omega\). This is maintained at temperature and concentration, \(T_e(1 + \varepsilon) f(t')\) and \(C_e(1 + \varepsilon) f(t')\) in which, \(T_e \gg 1, f(t'), C_e \gg 1\) where \(f(t')\) is an arbitrary function of time and \(\varepsilon\) is a parameter small. Following the arguments in Opara (1990) and employing equation 2.2b. The governing equations for a \(\Omega\) medium are:

\[
\frac{\partial u'}{\partial t} - 2\Omega v' = \nabla^2 u' - \frac{\mu^2 H^2}{\rho_\infty} u' + g\beta(T - T_e) + \gamma\delta(c - c_e) \tag{3}
\]
\[
\frac{\partial v'}{\partial t} + 2\Omega u' = \nabla^2 v' - \frac{\mu^2 H^2}{\rho_\infty} v' \tag{4}
\]
\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u')}{\partial x} = k_r \frac{\partial^2 T}{\partial z^2} - 4\sigma \alpha (T^4 - T_e^4) \tag{5}
\]
\[
\frac{\partial c}{\partial t} + \frac{\partial (\rho c u')}{\partial x} = D_r \frac{\partial^2 c}{\partial z^2} - K_r (c^4 - c_e^4) \tag{6}
\]

Where \( U = u' + \Omega v' = 0, T = T_e(1 + \varepsilon) f(t'), \) and \( C = C_e(1 + \varepsilon) f(t') \) on \(Z = 0, u' = 0 \)
\( c' = 0, v' = 0, T = T_\infty, C = C_\infty \) on \(Z \to \infty \)

\[ \theta = \text{characteristic velocity} \]
\[ \kappa_r = \text{thermal conductivity} \]
\[ g = \text{gravitational acceleration} \]
\[ \sigma_c = \text{electrical conductivity} \]
\[ \alpha, \beta = \text{the coefficients of volumetric expansion for concentration and temperature.} \]
\[ \epsilon = \text{small parameter} \]
\[ \nu = \text{kinematic viscosity.} \]
\[ \Omega = \text{angular velocity} \]
\[ \rho_{\infty} = \text{reservoir density.} \]

**Equation (3) to (7) can be written as**

\[
\frac{\partial \rho}{\partial t} + 2\Omega \rho u = \frac{\partial^2 \rho}{\partial z^2} - \nabla^2 \rho \tag{9}
\]
\[
\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial z^2} - R_T (\theta^4 - 1) \tag{10}
\]
\[
\frac{\partial c}{\partial t} + \frac{\partial (\rho c u)}{\partial x} = D_r \frac{\partial^2 c}{\partial z^2} - K_r (c^4 - c_e^4) \tag{11}
\]

**such that**

\[ \mathcal{Z} = 0; \quad q = 0; \theta = \theta_\infty[1 + \varepsilon \mathcal{H}(t)], \quad C = C_\infty[1 + \varepsilon \mathcal{H}(t)], \quad \mathcal{Z} \to \infty; q = 0; \theta = 1, C = 1 \]

and \( q = \mathcal{U} + i\mathcal{V}, \quad t = \sqrt{-1} \cdot \mathcal{E} \) is the rotational parameter. \( \mathcal{M}^2 \) is the magnetic parameter \( G_c \) and \( G_T \) are the free convection parameters for concentration and temperature respectively. \( R_T \) is the total radiation parameter. The mathematical statement of the problem embodies the solution of equation (9), (10) and (11) subject equation (12).

### IV. METHOD OF SOLUTION

The problem posed in equation (9), (10) and (11) very non-linear and generally will involve a step by step numerical integration by (say) the explicit finite difference scheme. However, if \( \varepsilon \) is small analytical solution could be possible by adopting regular perturbation. A great physical
insight could be shed on the whole of the CR transport by such
a perturbation scheme and this is the problem we shall pursue
in this investigation. We write:

\[ \Theta = \Theta^{(0)}(Z) + \varepsilon \Theta^{(1)}(Z, t) + \cdots \]  
\[ C = C^{(0)}(Z) + \varepsilon C^{(1)}(Z, t) + \cdots \]  
\[ q = q^{(0)}(Z) + \varepsilon q^{(1)}(Z, t) + \cdots \]  

Substituting equation (13) into equation (9)—(11) we have
the sequence of approximations equations.

\[ \frac{d^2 q^{(0)}}{dz^2} - \mathcal{M}^2 q^{(0)} + G_r(\Theta^{(0)} - 1) + G_c(C^{(0)} - 1) = 0 \]  
\[ \frac{d^2 \theta^{(0)}}{dz^2} - 4 \mathcal{R}_\theta(\Theta^{(0)} - 1) = 0 \]  
\[ \frac{d^2 C^{(0)}}{dz^2} - 4k_r(C^{(0)} - 1) = 0 \]  
\[ Z = 0; \theta^{(0)} = \theta_x; C^{(0)} = C_x; q^{(0)} = 1; Z \to \infty; \theta^{(0)} = 1; q^{(0)} = 0 C^{(0)} = 1 \]

and

\[ \frac{\partial q^{(1)}}{\partial t} + 2iE q^{(1)} = \frac{\partial^2 q^{(1)}}{\partial z^2} - \mathcal{M}^2 q^{(1)} + G_r \theta^{(1)} + G_c C^{(1)} \]

In equation (21) we assume

\[ t > 0. \text{ Thus by asymptotic approximation,} \]

the problem now splits into a steady flow on which is
superimposed a first order transient component, the solution
for \( \Theta^{(0)} \) and \( C^{(0)} \) are given.

\[ Z = \sqrt{\frac{5}{2 \mathcal{R}_\theta}} \int_{\Theta^{(0)}}^{C^{(0)}} ds = \frac{(s^5 - 5 s^4 + 4)}{3} \]  
\[ Z = \sqrt{\frac{1}{2 \mathcal{R}_r}} \int_{C^{(0)}}^{C} ds = \frac{(s^5 - 5 s^4 + 1)}{3} \]

and

\[ q^{(0)} = \exp \left[ - (\mathcal{M}^2 + 2iE) \frac{1}{2} Z \right] - \sqrt{\frac{5}{2 \mathcal{R}_r}} \int \Theta^{(0)} C^{(0)} dZ \]

\[ \sinh \left[ \frac{\mathcal{M}^2 + 2iE}{2} Z (\Theta^{(0)}, C^{(0)}) - Z(s) \right] (s - 1) ds \]

To solve equation (18), (19) and (20) and also consider the
boundary conditions in equation (21) we take the Laplace
transform with respect to time. Denoting the transformed
variable by \( \xi \) and placing a tilde over the transformed
function, the equation satisfied by \( \tilde{q}^{(1)}, \tilde{\theta}^{(1)}, \text{ and } \tilde{C}^{(1)} \) are;

\[ \frac{d^2 \tilde{q}^{(1)}}{d\xi^2} - (4 \mathcal{R}_\theta \Theta^{(0)2} + \zeta) \tilde{\theta}^{(1)} = 0 \]
\[ \frac{d^2 \tilde{C}^{(1)}}{d\xi^2} - (4 \mathcal{R}_r C^{(0)2} + \zeta) \tilde{C}^{(1)} = 0 \]

with boundary conditions

\[ Z = 0, \tilde{\theta}^{(1)} = \theta_x, \tilde{C}^{(1)} = C_x \text{ and } \tilde{q}^{(1)} = -[G_r \tilde{\theta}^{(1)} + G_c \tilde{C}^{(1)}] \]

\[ Z \to \infty, \tilde{\theta}^{(1)} = 0, \tilde{C}^{(1)} = 0 \]

The solution for \( \tilde{\theta}^{(1)} \) and \( \tilde{C}^{(1)} \) are

\[ \tilde{\theta}^{(1)} = \exp \left[ - (4 \mathcal{R}_\theta + \frac{1}{2} \mathcal{R}_r) \frac{1}{2} Z \right] \]
\[ \tilde{C}^{(1)} = \exp \left[ - (4 \mathcal{R}_r + \frac{1}{2} \mathcal{R}_r) \frac{1}{2} Z \right] \]

Hence, taking the inverse Laplace transform

\[ \theta^{(1)} = \frac{1}{2} e^{-2 \mathcal{R}_\theta Z} \text{erfc}\left[ \frac{Z}{2 \mathcal{R}_r} \right] + e^{2 \mathcal{R}_r Z} \text{erfc}\left[ \frac{Z}{2 \mathcal{R}_r} \right] \]
\[ C^{(1)} = \frac{1}{2} e^{-2 \mathcal{R}_r Z} Z \text{erfc}\left[ \frac{Z}{2 \mathcal{R}_r} \right] + e^{2 \mathcal{R}_r Z} \text{erfc}\left[ \frac{Z}{2 \mathcal{R}_r} \right] \]

Next we consider when \( \theta_x, C_x \) are arbitrary and approximate
\( \Theta^{(0)} \) \( C^{(0)} \) by

\[ \theta^{(0)2} = (\Theta_x^2 - 1) e^{-2 \beta Z} + 1 \]
\[ C^{(0)2} = (C_x^2 - 1) e^{-2 \gamma Z} + 1 \]

By virtue of equation (32) and (33) the solution for \( \theta^{(1)} \) and
\( C^{(1)} \) now reduces to

\[ \frac{\theta^{(1)}}{\theta_x} = \frac{I_2}{\xi_2} \frac{1}{(4 \mathcal{R}_\theta + \frac{1}{2} \mathcal{R}_r) \mathcal{R}_r + \xi_2} = \frac{I_1}{\xi_1} \frac{1}{(4 \mathcal{R}_r + \frac{1}{2} \mathcal{R}_r) \mathcal{R}_r + \xi_2} \]
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\[ c^1_k = \frac{J}{(4\pi T + \xi)^{3/2}} e^{\frac{\kappa}{(4\pi T + \xi)^{3/2}}} \]

and

\[ \eta = \mathcal{R}_T(\sqrt{T} - 1) \]

and

\[ I_n(x) \text{ and } l_n(x) \text{ are the Bessel and Modified Bessel function of the first kind respectively.} \]

Equation (34) and (35) have simple pole at \( \xi = 0 \) and another branch at \( \xi = 4\mathcal{R}_T \) and \( \mathcal{R}_T \) respectively. These Equations could then be inverted in the (39) and Bromwich contour with a suitable branch cut and we obtain

\[ \theta^{(1)} = \frac{l_n(x) e^{-\beta Z}}{14\pi \mathcal{R}_T^2 \eta} + \frac{e^{-4R_T \xi}}{2\pi i} \left\{ \int_0^\infty \frac{e^{-\xi t}}{(x + 4\mathcal{R}_T) l_n(t)} - \int_0^\infty \frac{e^{-\xi t}}{(x + 4\mathcal{R}_T) l_n(t)} \right\} \]

Because of the complexity of the above equations (36) and (37) we find it expedient to consider limiting values with

\[ I_n(x) \sim \frac{1}{(2\pi n)^{3/2}} e^{\frac{x}{2\pi n}}, \quad n \to \infty \]

We can show that

\[ \theta^{(1)} \sim \exp[-\frac{\beta Z (4\mathcal{R}_T + \xi)^{3/2}}{\xi}] \]

and

\[ c^{(1)}_k \sim \frac{1}{\mathcal{R}_T^{3/2}} e^{-2\frac{\beta Z^2}{(2t)^2}} \text{erfc} \left[ \frac{\beta Z}{(2t)^2} - (4\mathcal{R}_T t)^{3/2} \right] \]

We get

\[ q^{(1)} = \frac{1}{2} \int_0^\infty e^{-s(z-t)} \theta(z) \, dz \]

where

\[ S^2 = \xi + M^2 + 2tE, \]

We get

\[ q^{(1)} = \frac{1}{2} \int_0^\infty e^{-s(z-t)} \theta(z) \, dz \]
by convolution theorem we can deduce that.

\[ q^{(1)}(t) = \frac{1}{2\pi^2} \int_0^T \frac{1}{e^{-(\lambda^2/4\pi)^2}} e^{-((t^2 + 2\xi t)\tau - (\xi - \frac{t}{\tau})^2/4\tau)} dx \]

The solution is now complete.

Fig 1: plot of variation of concentration and Temperature with $Z$ for $R$ is 1.0

Fig 2: plot of variation of concentration and Temperature with $Z$ for $R$ is 0.5

Fig 3: plot of variation of concentration and Temperature with $Z$ for $R$ is 0.25

V. RESULTS AND DISCUSSIONS.

The problem of the effect of radiative heat transfer on cosmic rays (CR) in rotating cloudy interstellar medium has been solved making fairly realistic assumptions, for a small time-dependent perturbation of the temperature and adiabatic concentration, the non-linear problem is tackled by asymptotic approximation, giving solutions for steady state flow on which a first order transient component is superimposed.

Equations (22) and (23) give solutions for the steady-state components of the temperature $\theta^{(0)}$ adiabatic concentration $C^{(0)}$ and velocity $q^{(0)}$ fields. Equations (22) have been evaluated by numerical integration using three values of $R_T (0.5, 1.0$ and 1.5 to show the dependence of $\theta^{(0)}, C^{(0)}$ on $Z$. These are shown in figure 1, 2, and 3. It can be seen from these curves that their cube values say approach unity asymptotically. Furthermore, when the parameters are $\gamma = 10$ and $\beta = 12$, the variation of the adiabatic concentration $C^{(0)}$ and temperature $\theta^{(0)}$ with $Z$ for steady state flow are given by equations (32 and 33) These are almost identical to that deduced from equation (22a) and (22b). They are illustrated by the red line dashed curves in figures 1, 2 and 3.

Also for the case where $\theta^{(0)} = 1$ the transient component of the temperature exhibit a standing wave structure. The magnitude of the standing wave increases intensely when the solute concentration $C^{(0)} = 1$. This means that for $\theta^{(0)} = 1 = C^{(0)}$, the intense radiative heat is large, that is when $R_T$ is large. There is contrary to a relatively low radiation when $R_T$ is of order 0(1) in which the standing wave structure is no longer exhibited by both the temperature and concentration fields. These are shown in equations (41) - (43). However, equation (49) give a complete solution of the transient component of the velocity field.

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REFERENCES