Research on fractal structure of generalized J set utilized compound complex map

Wang Chun-mei, Hu Chun-hua

Abstract—**This paper generalizes Entwistle's complex map as $f(z): z \leftarrow (z^{\alpha}+c)^{\beta}+c(\alpha,\beta \in R)$, constructs a set of generalized Julia sets(J sets) by using the escape time. Adopting the experimental mathematics method of combining the analytic function with computer aided drawing, this paper researches on the structure topological inflexibility and the discontinuity evolution law of the generalized J sets. The researches as below: ① generalized J sets have $\alpha-$fold rotation symmetry and its center is the origin when α is integer;②the different choices of angle lead to the different evolution of generalized J sets.**

***Index Terms*—generalized Julia sets; complex plane; escape time algorithm;**

I. Introduction

In recent 20 years, scholars have made deep researches on the M-J sets generated from the complex map $z \leftarrow z^{\alpha}+c(\alpha=2)$ [1-2]. Based on these, Lakhtakia[3] and Gujar[4,5] have explored the structure of the generalized M-J sets for $\alpha \in R$; Glynn[6] has discovered symmetrical evolution of generalized M sets when angle $\theta \in [-\pi,\pi)$; Dhurandhar[7] et al have discussed fractal structures of J sets when $\alpha<0$; Author has explored the structure topological inflexibility and the discontinuity evolution law of generalized M sets[8]; Sasmor[9] has analysed fission evolution law of generalized M-J sets when angle $\theta \in [-\pi,\pi)$ and α is rational number; Romera et al have explored the nesting relationship of "petal" of generalized M sets at "Misiurewicz"[10,11]; Geum[12] and Author[13] have researched the structure and distributing of the periodicity "petal" and topological law of periodicity orbits of the generalized M sets.Expand or transform the complex map: Lakhtakia has researched on the switched processes of J sets[14]; Michelitsch has researched the J sets constructed by a simple non-analytic complex map[15]; Shirriff has presented the methods of combination M sets; Chen[17] et al have expanded the complex map as $z \leftarrow z^{w}+c(w \in R)$. Wang X Y[18,19] et al has generalized the work of Michelitsch[15] and Shririff[16], has studied combination generalized M sets with real exponential map and structure of generalized J sets of non-analytic complex map. In addition, Entwistle has studied the structure of generalized J sets of compound complex map. $z \leftarrow (z^2+c)^2+c$ This paper generalizes the complex map as $z \leftarrow (z^{\alpha}+c)^{\beta}+c(\alpha,\beta \in R)$, researches the fractal structures of generalized J sets and discontinuity evolution law, and compute the Hausdorff distance of two generalized J sets.

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II. Theory and Methods

For $f(z)=z^{\alpha}+c(\alpha \in R)$, if ω satisfies $f(\omega)=\omega$, then ω is called fixed point of f. If there is the minimum positive integer p and p satisfies $f^{p}(\omega)=\omega$, we say that ω is *p*-periodicity point of f. If complex differential quotient $(f^{p})'(\omega)=\lambda$, and $|\lambda|>1$, then we call the point ω is repelling. From the famous Montel theorem, we know that J_f which is the generalized J sets of f is the closure consisting of exclusive periodic points [21]. If $c=0$, then $f(z)=z^{\alpha}$ and $f_k(z)=z^{\alpha^k}$, and the points which satisfy $f^{p}(\omega)=\omega$ are $\left\{\exp(\frac{2\pi iq}{\alpha^{p}-1}):0 \le q<\left|\alpha^{p}-1\right|-1\right\}$.

If let $|\alpha|>1$,then these points satisfy $\left|(f^{p})'(z)\right|=\left|\alpha^{p}\right|>1$. So these points are repelling and J_f is the unit circle $|z|=1$ .It is obvious that when $k \to \infty$, if $|z|<1$, then $f^{k}(z) \to \infty$ or 0; If $|z|>1$, $f^{k}(z) \to \infty$ or 0; But if $|z|=1$, $f^{k}(z)$ is always on J_f. When iterated, J_f is the bondary of the point sets which converge to 0 and ∞ separately. It is certain that J_f is not fractal in such special case. If c is the small complex number, then $f(z)=z^{\alpha}+c$. It is easy to make out that if z is also small, then $f^{k}(z) \to \omega$ or ∞, where ω is the fixed point near by the origin. But if z is a great number, then $f^{k}(z) \to \omega$ or ∞. Now it appears that J_f is the fractal curve, though J_f is the boundary of two different kinds of sets.

Definition 1 If $f: z \leftarrow z^{\alpha} + c(|\alpha| > 1)$ is the complex map on the Riemann globe $\hat{C}$, F_f represents the set of the complex quantity z whose trajectory is limited, i.e.,

$$F_f = \{z \in C: \{|f^k(z)|\}_{k=1}^{\infty} \ is\ limited\}$$

then the sets are called filled generalized J sets corresponding to the complex map f, which denoted by J_f, i.e.,

$$J_f = \partial F_f$$

The iteration should begin with the critical point of f if the generalized Mandelbrot sets is constructed from the complex map $f: z \leftarrow z^{\alpha} + c(\alpha \in R)$. When $\alpha > 1$, the critical point of f is ∞. If we let $z_0 = \infty$, then $z_1 = c, z_2 = c^{\alpha} + c, \ldots$. So in order to avoid overflow, the initial iterating point is chosen as $z_0 = c$. But what should be noticeable is that if c is chosen as the initial point when $\alpha \in [0,1]$, the images obtained are not the real M sets. The reason is that f does not have any critical points when $\alpha = 1$, so there is no point in discussing the trajectory of the critical point; when $0 \le \alpha < 1$, the critical point is ∞, the parameter c is not on the trajectory of ∞; therefore, the resulting image of the iteration from c as the initial point are not the M sets actually.

III. Experiment and Result

3.1. $\alpha = \pm\eta$ and $\beta = \pm\delta$

Fig.1 shows the typical generalized J sets for integer value of α, β and complex constant $c = 0.5 + 0.5i$ generated by compound complex map. The generalized J sets for $\alpha = \eta$ and $\beta = \delta$ resemble a flower consisting of η major petals, and chaos region is embedded in the stable region; when $\alpha = -\eta$ or $\beta = -\delta$, the generalized J sets resemble η secondary planets encircling the origin, and chaos region is embedded in the stable region. When $\alpha = +\eta$ and $\beta = -\delta$, the generalized J sets have α near-circular shapes inside, while have $\alpha \times |\beta|$ outside.

Theorem 1 If the generalized J sets are generated by the complex map $f: z \leftarrow (z^{\alpha} + c)^{\beta} + c\ (\alpha, \beta \in R)$, then

$$f^k(z) = f^k(ze^{i\frac{2\pi j}{\alpha}})\ (1 \le k \le N; 0 \le j < |\alpha| - 1).$$

Proof

$\because$

$$f(ze^{i\frac{2\pi j}{\alpha}}) = [(ze^{i\frac{2\pi j}{\alpha}})^{\alpha} + c]^{\beta} + c$$
$$= [z^{\alpha} e^{i(2\pi j)} + c]^{\beta} + c = (z^{\alpha} + c)^{\beta} + c = f(z)$$

Suppose $f^k(z) = f^k(ze^{i\frac{2\pi j}{\alpha}})$ is tenable.

As well as $\because$

$$f^{k+1}(ze^{i\frac{2\pi j}{\alpha}}) = f^k[f(ze^{i\frac{2\pi j}{\alpha}})] = f^k[f^1(z)]$$
$$= f^{k+1}(z)$$

So according to Eq.(1), we can know that this proposition is tenable. Theorem ! states that the generalized J sets for $\alpha = \pm\eta$ have $\eta -$fold rotation symmetry and its center is origin.

The fractal structures of the generalized J sets for positive integer values of α can be explained as following, J_f is the closure composed of repelling periodic points of the complex map f, the poins on major petals tend towards the fixed point ω approaching the origin after one iterate; As F_f has rotating symmetric structure and its center is the origin, let us suppose the center of major petal z_1 get to the origin after one iterate, i.e., $f^1(z_1) = 0$. L_1 indicates the major petal.

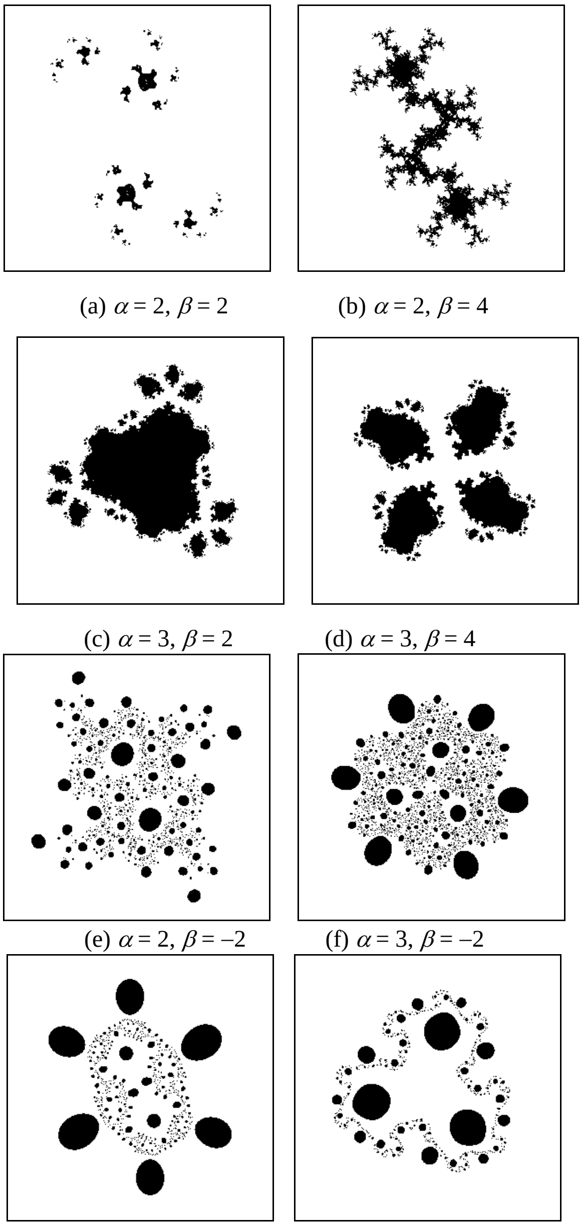
Theorem 2. when $\alpha = \eta$, the generalized J sets generated by complex map $f: z \leftarrow (z^{\alpha} + c)^{\beta} + c(\alpha, \beta \in R)$ have η major petals L_1, and the polar coordinate of center of major petals c is $(-c + (-c)^{\frac{1}{\beta}})^{\frac{1}{\alpha}}$.

Prove: suppose the center of L_1 is z_1, so $f^1(z_1) = 0$, $(z_1^{\alpha} + c)^{\beta} + c = 0$, so

$$z_1 = (-c + (-c)^{\frac{1}{\beta}})^{\frac{1}{\alpha}}. \qquad (1)$$

When $\alpha = \eta$, we can know that from (1), there are η major petals and the angle of the centers of any two L_1 is $2\pi/\eta$, the proposition is tenable. When $\alpha = \eta$, $\beta = \delta$ and $c = 0.5 + 0.5i$, we can extract z_1 from (1), and $|z_1|$ increases with α increasing, tend towards to 1 at end. Compared with Fig.1(a)~1(h), Theorem 2. goes along with generalized J sets, consequently proves the proposition.

Fig.1 shows that there are small petals on major petals, and there are smaller petals on small petals,…, such overlapping embedment structure appears on different levels. The centers of the biggest petal on major petal reaches the center of major petal reaches the center of major petal after one iterate, and then reach the origin after another iterate. The rest may be deduced by analogy, if L_1 represents major petal, then certain small petal whose center z_k satisfies $f^k(z_k) = 0(2 \le k \le N)$ can be represented as L^k.



(a) $\alpha=2$, $\beta=2$ (b) $\alpha=2$, $\beta=4$

(c) $\alpha=3$, $\beta=2$ (d) $\alpha=3$, $\beta=4$

(e) $\alpha=2$, $\beta=-2$ (f) $\alpha=3$, $\beta=-2$

(g) $\alpha=2$, $\beta=-3$ (h) $\alpha=3$, $\beta=-3$

Fig.1 The generalized Julia sets of α and β are integer.

According to $f:(z^\alpha+c)^\beta+c\ (\alpha=\eta,\beta=\delta)$, we get its converse map $f^{-1}: z\leftarrow((z-c)^{\frac{1}{\beta}}-c)^{\frac{1}{\alpha}}$ $(\alpha=\eta,\beta=\delta)$, so L^k can be represented as

∴

$$\mathrm{L}^k=\underbrace{f^{-1}f^{-1}\cdots f^{-1}}_{(k-1)次}\{\mathrm{L}^1\}=\underbrace{f^{-1}f^{-1}\cdots f^{-1}}_{(k-2)次}\{\mathrm{L}^2\}. \quad (2)$$

f^{-1} has $\alpha\beta$ complex roots, and the number of L^1 is α, so the number of L^k after f^{-1} iterate (k–1) times is $\alpha^k\beta^{k-1}$.

$\forall z\in\mathrm{L}^k$, $\exists f^1(z)\in\mathrm{L}^{k-1}$. If suppose the center of L^k is z_k, and the center of $\mathrm{L}^{k\text{-}1}$ is z_{k-1}, then $f^1(z_k)=z_{k-1}$. Considering L^2 first, the center of L^1 is image of the center z_2 of L^2, according to Theorem 2, we can get $z_1=(-c+(-c)^{\frac{1}{\beta}})^{\frac{1}{\alpha}}$, so $(z_2^\alpha+c)^\beta+c=(-c+(-c)^{\frac{1}{\beta}})^{\frac{1}{\alpha}}$. We can get that:

$$z_2=(((-c+(-c)^{\frac{1}{\beta}})^{\frac{1}{\alpha}}-c)^{\frac{1}{\beta}}-c)^{\frac{1}{\alpha}}$$

Equally, the center coordinate of L^3 is

$$z_3=(((((-c+(-c)^{\frac{1}{\beta}})^{\frac{1}{\alpha}}-c)^{\frac{1}{\beta}}-c)^{\frac{1}{\alpha}}-c)^{\frac{1}{\beta}}-c)^{\frac{1}{\alpha}} \quad (3)$$

The rest may be deduced by analogy, therefore, from theoretical view, when $N\to\infty$, the generalized J sets for positive integer values of α and β have infinite overlapping embedment self-similar geometry structure.

Theorem 3. when $\alpha=+\eta$, $\beta=-\delta$, the generalized J sets generated by complex map $f: z\leftarrow(z^\alpha+c)^\beta+c(\alpha,\beta\in R)$ have α near-circular shapes inside the near-round which center is origin, and radius is $R^{\frac{1}{\alpha\beta}}$.

Proof: for $\alpha>0$ 、 $\beta<0$, if $\|x\|<1$, therefore $\|f^1(x)\|\geq R$. According to Theorem 1. and the structure of the generalized J sets for $\alpha=+\eta$ 、 $\beta=-\delta$, we can consider that any point of the α near-circular satisfies $\|f^1(z)\|\geq R$. Suppose $z=\|z\|e^{i\theta}$ ， $c=\|c\|e^{i\varphi}$, so

$$f^1(z)=(\|z\|^\alpha e^{i\alpha\theta}+\|c\|e^{i\varphi})^\beta+\|c\|e^{i\varphi}.$$

$$\because \alpha>0 、\|z\|>1 \text{ and } \|c\|=0.707, \therefore \|z\|^\alpha >> \|c\|.$$

According to (4), $\|f^1(z)\|\cong\|z\|^{\alpha\beta}$. $\because \|f^1(z)\|\geq R$ ， $\therefore \|z\|^{\alpha\beta}\geq R$. So we can deduce

$$\|z\|\leq R^{\frac{1}{\alpha\beta}}. \quad (5)$$

Compute square of the module of (4), has

$$\|f^1(z)\|^2\approx\|z\|^{2\alpha\beta}+2\|z\|^{\alpha\beta}\|c\|\cos(\alpha\beta\theta-\varphi)+\|c\|^2. \quad (6)$$

According to (6), we know that $\|z\|$ is function of θ. Based on (5), $\|z\|$ is represented

$$\|z\|=R^{\frac{1}{\alpha\beta}}[1-\delta(\theta)]. \quad (7)$$

$\delta(\theta)<<1$. Substituting Eq.(7) into Eq.(6), Based on $\|f^1(z)\|^2\geq R^2$ and take a first-order approximation, we get that

$$\delta(\theta)\cong\frac{\|c\|}{|\alpha\beta|R}\cos(|\alpha\beta|\theta-\varphi). \quad (8)$$

Obviously, the error of this near-round also has $\alpha\beta-$fold rotation symmetry, and the maximum of error appears when $\theta=\dfrac{2k\pi+\varphi}{|\alpha\beta|}$ (k is integer, and $0\leq k\leq|\alpha\beta|-1$).

In the process of prove of Theorem 3. we don't restrict the minus integer of α, so the conclusion of Theorem 3. is suitable to situation of α is minus decimal fraction.

Fig.1(i)~1(p) shows that there are small petals on major petals, and there are smaller petals on small petals,…, such overlapping embedment structure appears on different levels. Any point z in this near-round satisfies $\left\|f^1(z)\right\| \ge R$, if it is represented as B^1. then certain secondary planet whose any point z satisfies $\left\|f^k(z)\right\| \ge R$ and $\left\|f^{k-1}(z)\right\| < R$, so can be represented as B^k.

According to Eq.(2), f^{-1} has $|\alpha\beta|$ complex roots, and the number of B^1 is α, so the number of B^k after f^{-1} iterate $(k-1)$ times is $\alpha^k\left|\beta^{k-1}\right|$. From theoretical view, when $N \to \infty$, the generalized J sets for negative integer values of α and β have infinite overlapping embedment self-similar geometry structure.

3.2 $\alpha = \pm(\eta+\gamma)$ or $\beta = \pm(\delta+\varepsilon)$

The generalized J sets of $\alpha = \eta+\gamma$ and $\beta = \delta+\varepsilon$ are similar to an asymmetric flower consisting of η major petals and an embryonic petal. With the increase of γ, the embryonic petal increases in size and develops into a new major petal.(Fig.2(a)~2(d),Fig.2(e)~2(h)). The generalized J sets of $\alpha = -(\eta+\gamma)$ and $\beta = -(\delta+\varepsilon)$ consist of η major satellite structures and an embryonic satellite structure increases in size and develops into a new major satellite structure with the increasing of γ from 0 to 1(Fig.2(e)~2(l),Fig.2(m)~2(p)).

DeMoivre's theorem is used to calculate z^α and $(z^\alpha+c)^\beta$, for instance

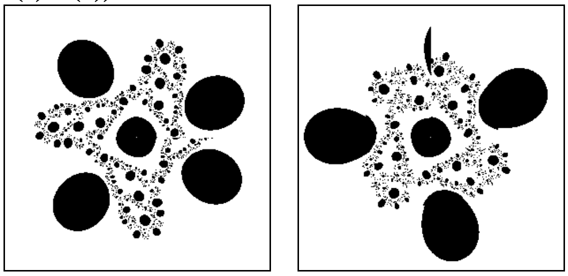
$$(z^\alpha+c)^\beta = \left\|z^\alpha+c\right\|^\beta(\cos\beta\theta + i\sin\beta\theta) \quad (9)$$

This involves the choice of the principal range of the phase angel θ. The authors choose four types as follows: [0, 2π),[−3π/2, π/2),[−π, π) and [−π/2, 3π/2). When $\beta = \pm\eta$, the use of Eq.(9) will not be affected because

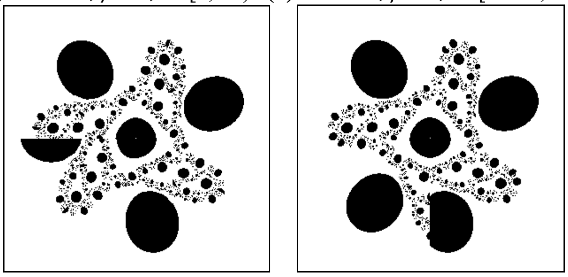
$$\begin{cases}\cos(\beta\theta) = \cos(\beta\theta + 2\pi\beta)\\ \sin(\beta\theta) = \sin(\beta\theta + 2\pi\beta)\end{cases}. \quad (10)$$

But when $\beta = \pm(\delta+\varepsilon)$, Eq.(10) is not valid. So the different choice of the principal range for θ will give rise to different evolutions of the generalized J sets. Besides, when using Eq.(10), if the argument $\beta\theta$ goes beyond the above the four ranges above, the argument $\beta\theta$ will be adjusted by adding or subtracting the integer times of 2π, which results in the discontinuity and collapse of the generalized J sets and appears the embryonic petal. The argument θ lies in following ranges: [0, 2π),[−3π/2, π/2),[−π, π) and [−π/2, 3π/2). This lead to the appearance of the embryonic petal, but only appears at the positive real axis, the positive imaginary axis, the negative real axis or the negative imaginary axis where the argument θ is not continuous.

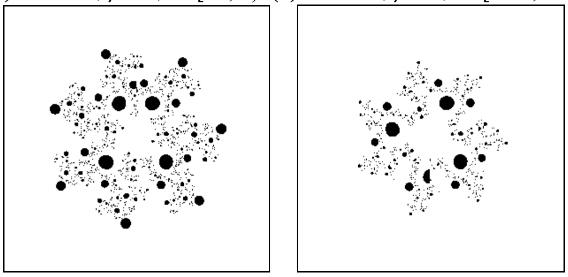
Aside, according to Theorem 1. the generalized J sets also have $\eta-$fold rotation symmetry(Fig.3). But according to (10): when α is constant, the different choice of angle θ, the value of $(z^\alpha+c)^\beta$ is also different, so this will lead to the different evolvement of generalized J sets.(Fig.3(a)~3(d), Fig.3(e)~3(h)).



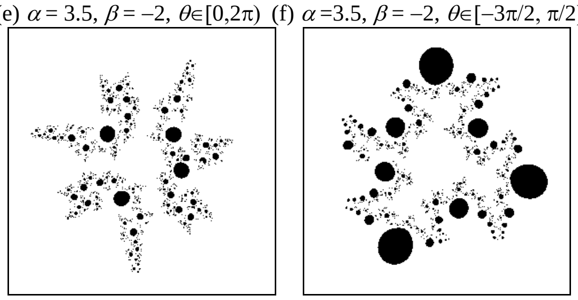
(a) $\alpha = -3.5$, $\beta = 2$, $\theta \in [0, 2\pi)$ (b) $\alpha = -3.5$, $\beta = 2$, $\theta \in [-3\pi/2, \pi/2)$



(c) $\alpha = -3.5$, $\beta = 2$, $\theta \in [-\pi, \pi)$ (d) $\alpha = -3.5$, $\beta = 2$, $\theta \in [-\pi/2, 3\pi/2)$

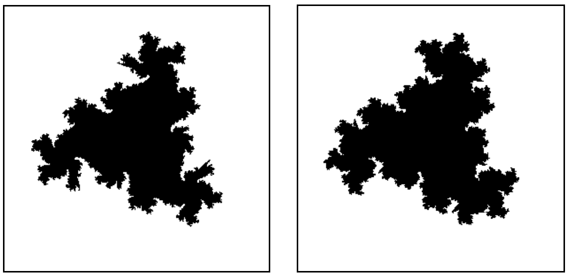


(e) $\alpha = 3.5$, $\beta = -2$, $\theta \in [0, 2\pi)$ (f) $\alpha = 3.5$, $\beta = -2$, $\theta \in [-3\pi/2, \pi/2)$

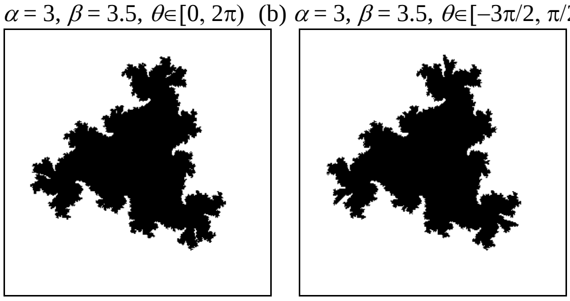


(g) $\alpha = -3.5$, $\beta = -2$, $\theta \in [-\pi, \pi)$ (h) $\alpha = -3.5$, $\beta = -2$, $\theta \in [-\pi/2, 3\pi/2)$

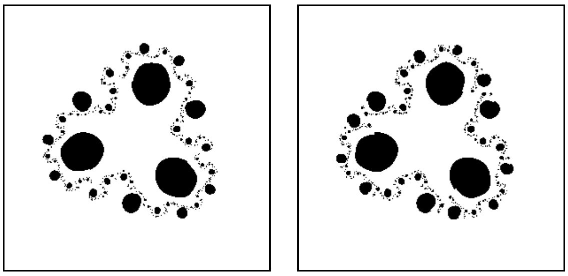
Fig.2 the generalized Julia sets of $\alpha = \pm(\eta+\gamma)$，$\beta = \pm\delta$



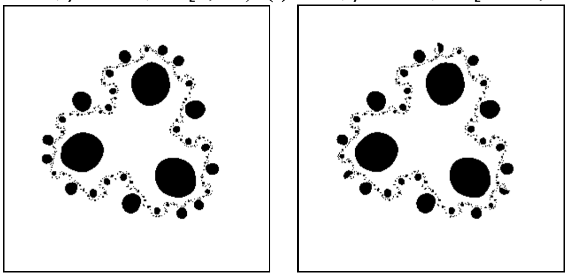
(a) $\alpha = 3$, $\beta = 3.5$, $\theta \in [0, 2\pi)$ (b) $\alpha = 3$, $\beta = 3.5$, $\theta \in [-3\pi/2, \pi/2)$



(c) $\alpha = 3$, $\beta = 3.5$, $\theta \in [-\pi, \pi)$ (d) $\alpha = 3$, $\beta = 3.5$, $\theta \in [-\pi/2, 3\pi/2)$



(e) $\alpha = 3$, $\beta = -3.5$, $\theta \in [0, 2\pi)$ (f) $\alpha = 3$, $\beta = -3.5$, $\theta \in [-3\pi/2, \pi/2)$



(g) $\alpha = 3$, $\beta = -3.5$, $\theta \in [-\pi, \pi)$ (h) $\alpha = 3$, $\beta = -3.5$, $\theta \in [-\pi/2, 3\pi/2)$

Fig.3 the generalized Julia sets of $\alpha = \eta$, $\beta = \pm(\delta + \varepsilon)$

IV. CONCLUSION

This paper generalizes Entwistle's complex map as $z \leftarrow (z^{\alpha} + c)^{\beta} + c(\alpha, \beta \in R)$, constructs a set of generalized Julia sets(J sets)by using the escape time. Adopting the experimental mathematics method of combining the analytic function with computer aided drawing, this paper researches on the structure topological inflexibility and the discontinuity evolution law of the generalized J sets. The researches as below: ①generalized J sets have $\alpha -$fold rotation symmetry and its center is the origin when α is integer;②the different choices of argument lead to the different evolution of generalized J sets.

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