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On Interest Rate Option Pricing with Jump Processes

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Abstract— **In this study, we investigate the pricing of interest rate options in three arbitrage-free models with jump process which are Vasicek and Cox-Ingersoll-Ross (CIR) models of stochastic interest rate and Heath-Jarrow-Morton (HJM) model for stochastic forward rate. Solutions of Hull and White (HW) type model with jump are derived directly using a system of differential equations and the relationship between short rate and forward rate processes which is obtained under the extended restrictive condition on jump and volatility can be used to have the formula of bond price. We also analyse the option values of three proposed jump models obtained by Monte Carlo simulations.**

Index Terms—**Stochastic Interest Rate Option, Jump Process, HJM model, Monte Carlo simulation.**

I. INTRODUCTION

In pricing and hedging with financial derivatives, term structure models with jump are particularly important [8], since ignoring jumps in financial prices may cause inaccurate pricing and hedging rates [1]. Solutions of term structure model under jump-diffusion processes are justified because of movements in interest rates displaying both continuous and discontinuous behaviors [3]. Moreover, to explain term structure movements used in the latent factor models, it means how macro variables affect bond prices and the dynamics of the yield curves [2]. Current research using jump-diffusion processes relies mostly on two classes of models: the affine jump-diffusion class [7] and the quadratic Gaussian [5]. We consider the classes of Hull and White (HW) model with jump and Heath-Jarrow-Morton (HJM) model based on jump to investigate solutions for interest rate option price on the proposed models. In this paper, we show the actual proof analysis of the HJM model based on jump easily under the extended restrictive condition of Ritchken and Sankarasubramanian (RS) [10]. By beginning with certain forward rate volatility processes, it is possible to obtain classes of interest models under HJM model based on jump that closely resembles the traditional models [6].

Finally, we confirm that there is a difference between interest rate option prices which are obtained by HW model with jump and HJM model based on jump through the empirical computer simulation which used Monte Carlo simulation (MCS), which is used by many financial engineers to place a value on financial derivatives. The structure of the remainder of this paper is as follows. In Section 2, investigate the pricing of bonds on arbitrage-free models with jump. In Section 3, the pricing of interest rate option caplets on arbitrage-free models with jump are presented. Section 4 explains the simulation procedure of the proposed models using Monte Caro simulation and the proposed models' performances are evaluated based on simulations. Finally, Section 5 concludes this paper.

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II. PRICING OF BOND ON STOCHASTIC INTEREST MODELS WITH JUMP

All our models will be set up in a given complete probability space (Ω, F_t, P) and an argument filtration $(F_t)_{t\geq 0}$ generated by the Wiener process $W(t)$ and the Poisson process $N(t)$ with intensity rate λ, which represents the total number of extreme shocks that occur in a financial market until time t [9]. If there is one jump during the period $[t, t+dt]$ then $dN(t) = 1$, and $dN(t) = 0$ represents no jump during that period. In the same way that a model for the asset price is proposed as a lognormal random walk, let us suppose that the interest rate r and the forward rate f are governed by a Stochastic Differential Equation(SDE) of the form

$$dr = \mu_r(r,t)dt + \sigma_r(r,t)dW(t) + JdN(t), \qquad (1)$$

$$df(t,T) = \mu_f(t,T)dt + \sigma_f(t,T)dW(t) + JdN(t) \qquad (2)$$

where $\sigma(r, t)$ is the instantaneous volatility, $\mu_r(r, t)$ is the instantaneous drift, $\mu_f(t, T)$ represents drift function, $\sigma_f(t, T)$ is a volatility coefficient, $W(t)$ is the standard Wiener process, and jump size J is normally distributed, that is, $J \sim N(\mu, \gamma^2)$. When interest rates follow the SDE (1), a bond has a price of the form $V(t, T)$ at time t with maturity T; the dependence on T will only be made explicitly when necessary. To get the bond pricing equation with jump, we set up a riskless portfolio containing two bonds with different maturities T_1 and T_2 and then we applied the jump-diffusion version of Ito's lemma. Hence, we derive the partial differential equation (PDE) for bond pricing as below.

Theorem 1 *If r satisfy SDE (1), then the zero-coupon bond pricing equation with jumps is derived by*

$$\frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_r}^2\frac{\partial^2 V}{\partial r^2} + (\mu_r - \sigma_r\omega)\frac{\partial V}{\partial r} = rV - \lambda\mathbb{E}[V(r+J,t) - V(r,t)], \qquad (3)$$

where $\omega = \omega(r, t)$ is the market price of risk. The final condition corresponds to the payoff on maturity and so $V(T, T) = 1$. Boundary conditions depend on the form of $\mu_r(r, t)$ and $\sigma_r(r,t)$.

A. The Hull-White Model with Jump

Let be $V(t, T)$ the price at time t of a discount bond.
A solution of the form:

$$V(t,T) = \exp[-A(t,T)\, r_t + B(t,T)] \qquad (4)$$

can be assumed. We now consider a quite different type of random environment. In this paper, we extend a jump-diffusion version of equilibrium single factor model to reflect this time dependence. This leads to the following model for $r(t)$:

$$dr(t) = [\theta(t) - a(t)r(t)]dt + \sigma(t)r(t)^{\beta}dW(t) + JdN(t), \qquad (5)$$

where $\theta(t)$ is a time-dependent drift; $\sigma(t)$ is the volatility factor; $a(t)$ is the reversion rate; $W(t)$ is the standard Wiener process; and $N(t)$ is the Poisson process with intensity rate λ. We investigate the case $\beta = 0$ which is an extension of Vasicek's jump-diffusion model and the case $\beta = 1/2$ which is an extension of CIR jump-diffusion model. Under the process specified in equation (5), $r(t)$ is defined as:

$$r(t) = r_0(t)\left[r(0) + \int_0^t {r_0}^{-1}(u)\theta(u)du + \int_0^t {r_0}^{-1}(u)\sigma(u)r(u)^{\beta}dW(u) + \sum_{i=1}^{N(t)} {r_0}^{-1}(T_i)J_i\right] \qquad (6)$$

where $r_{0(t)} = \exp\left[-\int_0^t a(u)du\right] (= e^{-a(t)t}$ if $a(t)$ is constant), T_i is the time when i th jump happens, $0 < T_1 < T_2 < \cdots < T_{N(t)} < t$, and $N(t)$ is the number of jumps occurring during the period $[0, t]$. For the constant coefficient case, the conditional expectation and variance of jump-diffusion process given the current level are

$$\mathbb{E}[r(t)] = e^{-a(t)t}r(0) + \left(\frac{\theta(t)+\lambda\mu}{a(t)}\right)\left(1 - e^{-a(t)t}\right), \qquad (7)$$

and

$$\mathbb{V}ar[r(t)] = \begin{cases} \dfrac{\sigma(t)^2 + \lambda(\mu^2+\gamma^2)}{2a(t)}\left(1 - e^{2a(t)t}\right) & : \beta = 0, \\[2ex] \dfrac{r(0)\sigma(t)^2}{a(t)}\left(e^{-a(t)t} - e^{-2a(t)t}\right) & \\[2ex] + \dfrac{\sigma(t)^2(\theta(t)+\lambda\mu)}{2a(t)^2}\left(1 - e^{-a(t)t}\right)^2 & : \beta = \dfrac{1}{2}. \\[2ex] + \dfrac{\lambda(\mu^2+\gamma^2)}{2a(t)}\left(1 - e^{-2a(t)t}\right) & \end{cases}$$

To drive the pricing formula of bond on the Hull-White model with jump, we use a two-term Taylor's expansion theorem to represent the expectation terms of equation (3) is given by

$$\mathbb{E}[V(r+J,t) - V(r,t)] = \left[-\mu A(t,T) + \frac{1}{2}(\gamma^2+\mu^2)A(t,T)^2\right]V(r,t),$$

where a jump size $J \sim N(\mu, \gamma^2)$. Thus, we get the partial differential bond pricing equation:

$$[\theta(t) - a(t)r(t) - \omega(t)\sigma(t)r(t)^{\beta}]V_r + V_t + \frac{1}{2}\sigma(t)^2 r(t)^{2\beta}V_{rr} - rV + \lambda V\left[-\mu A(t,T) + \frac{1}{2}(\gamma^2+\mu^2)A(t,T)^2\right] = 0. \qquad (8)$$

By substituting the value of bond (4) into (8), we obtain the equations for $A(t, T)$ and $B(t, T)$.

Theorem 2 (Extended Vasicek Model : $\beta = 0$)

$$-\frac{\partial A}{\partial t} + a(t)A - 1 = 0 \qquad (9)$$

and

$$\frac{\partial B}{\partial t} - \phi(t)A + \frac{1}{2}\sigma(t)^2A^2 + \lambda\left[-\mu A + \frac{1}{2}(\gamma^2+\mu^2)A^2\right] = 0, \qquad (10)$$

where $\phi(t) = \theta(t) - \omega(t)\sigma(t)$ and all coefficients are constant.

Theorem 3 (Extended CIR Model : $\beta = \frac{1}{2}$)

$$-\frac{\partial A}{\partial t} + a(t)A + \frac{1}{2}\sigma(t)^2A^2 - 1 = 0 \qquad (11)$$

and

$$\frac{\partial B}{\partial t} - (\theta(t)+\lambda\mu)A + \frac{1}{2}\lambda[(\gamma^2+\mu^2)A^2] = 0, \qquad (12)$$

where all coefficients are constant.

From Theorem 2 and 3, to satisfy the final data that $V(T, T) = 1$ we must have $A(T, T) = 0$ and $B(T, T) = 0$.
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B. HJM model based on Jump

We denote as $f(t, T)$ the instantaneous forward rate at time t for instantaneous borrowing at time T ($\geq t$). Then the price at time t of a discount bond with maturity T, is defined as

$$V(t,T) = \exp\left(-\int_t^T f(t,s)ds\right). \qquad (13)$$

We consider the one-factor HJM model with jump under the corresponding risk-neutral measure Q, and we obtain the SDE given by

$$df(t,T) = \sigma_f^*(t,T)dt + \sigma_f(t,T)dW^Q(t) + JdN(t), \quad (14)$$

where $\sigma_f^*(t,T) = \sigma_f(t,T)\int_t^T \sigma_f(t,s)ds$. $W^Q(t)$ is a Wiener process generated by the risk-neutral measure Q, and $N(t)$ is a Poisson process with intensity rate λ. In a similar way as before, the conditional expectation and variance of the SDE (14) given the current level are

$$\mathbb{E}[f(t,T)] = f(0,T) + \mathbb{E}\left[\int_0^t \sigma_f^*(s,T)ds\right] + \mu\lambda t$$

and

$$\begin{aligned} \mathbb{V}ar[f(t,T)] &= \mathbb{E}[(f(t,T) - \mathbb{E}[f(t,T)])^2] \\ &= \int_0^t \sigma_f^2(s,T)ds + (\gamma^2 + \mu^2)\lambda t. \end{aligned}$$

In this study, we use the relationship between short rate and forward rate processes to obtain the formula of bond price under the extended restrictive condition on jump and volatility[10].

Theorem 4 *Let the volatility form of SDE (14) be*

$$\sigma_f(t,T) = \sigma(t)(r(t))^\beta \eta(t,T)$$

with a deterministic function $\eta(t,T) = \exp\left(-\int_t^T a(s)ds\right)$ *to obtain the relationship between forward rates and short rates (5) in term structure model. The equivalent model is derived as follows;*

$$\begin{aligned} f(0,T) &= r(0)\eta(0,T) + \int_0^T \theta(s)\eta(s,T)ds \\ &\quad - \int_0^T \sigma^2(s)\, r^{2\beta}(s)\, \eta(s,T) \int_s^T \eta(s,u)du\, ds. \end{aligned}$$

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III. PRICING OF INTEREST RATE OPTION CAPLETS ON ARBITRAGE-FREE MODEL WITH JUMP

A popular interest rate option in the financial market is an interest rate cap which is designed to provide insurance against the rate of interest on floating-rate note rising above a certain level known as the cap rate. The period to reset the interest rate equal to (forward) LIBOR rate is called as tenor usually 3 months. And its payoff occurs on the next reset day i.e. a tenor later. The payoff at time $T + \delta$ with a principal L and a cap rate R is defined by

$$L\delta \max(r(T) - R, 0) \quad (15)$$

where δ is a tenor and T is the reset time. From this payoff value, we derive the interest rate option formula on the condition of log-normality under the forward measure $P^{T+\delta}$. This leads to the following pricing result.

$$\begin{aligned} Cap &= L\delta\, \mathbb{E}[D(T+\delta)\max(r(T) - R, 0)] \\ &= L\delta\, V(0, T+\delta)\, \mathbb{E}^{T+\delta}[\max(R(T) - R, 0)] \\ &= L\delta\, V(0, T+\delta)\, [f(0,T)\Phi(d_+) - R\,\Phi(d_-)] \end{aligned}$$

where

$$d_+ = \frac{\log(f(0,T)/R) + \frac{1}{2}\sigma_f^2 T}{\sigma_f \sqrt{T}},$$

$d_- = d_+ - \sigma_f\sqrt{T}$ and $\Phi(x)$ is the standard normal cumulative function as follows.

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}}\, dy.$$

IV. COMPUTATIONAL RESULTS

In this section, we explain about the simulation procedure on the pricing of interest rate options on the arbitrage-free models with jump. The Monte Carlo Method is actually a very general tool and its applications are by no means restricted to numerical integration. To perform this method, we divide the time interval $[t, T]$ into m equal time steps of length Δt each. For small time steps, we can obtain the bond price by sampling n short and forward rates paths under the discrete version of the risk-adjusted SDEs (5) and (14).

We investigate the pricing of interest rate option caplets on the arbitrage-free models with jump. For the simulation of estimating interest rate option models with jump, the parameter values could be assumed as $R = 0.045$, $a = 0.0001$, $b = 0.04$, $\theta = a \times b$, $\sigma = 0.08$, $\lambda = 1.4$, $\gamma = 0.001$, $\mu = 0$, $r0 = 0.04$, $\delta = 0.25$, and $L = 1$. Furthermore we conduct three runs of $n = 10,000$ trials per each model and divide the time into $m = 100$ time steps.

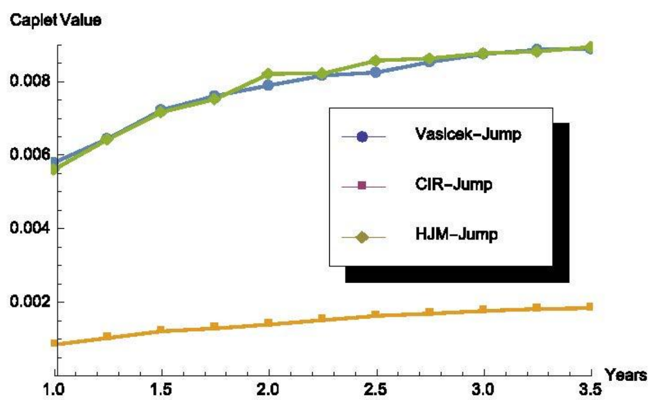

Figure 1: Pricing of interest rate option caplet with jump

Figure 1 represents the pricing of interest rate option caplets on arbitrage-free models using Monte Carlo Method, in which three interest rate models with jump, Vasicek, CIR, and HJM, show their characteristic behaviors. The value on CIR model is lower than those of two models, Vasicek and

HJM while the latter two models have very similar values even if they have different drift parameters. Every quarter year payoff time from 1 year to 3.5 years is selected in continuously comparative and increasing manner with 3 month tenor.

V. Conclusion

After investigating the models which allow the short term interest and the forward rate following a jump-diffusion process, we obtained the solutions on Hull-White jump models, which are more useful to evaluate the accurate estimate for the values of interest rate options in the financial market. Through Monte Carlo simulation of these solutions with jump, the prices of interest rate options on Vasicek-jump and HJM-jump models with short and forward rates give the similar figure which seems to go to the some limit as the maturity term increases while the graph of interest rate option caplet on the CIR-Jump model with the short term interest rate is closer to the option value of Black-Caplet formula without jump. The use of constant coefficients in dynamics of stochastic differential equations on jump processes sometimes leads to the difficulty in showing the difference of accuracy of financial modeling even if it gives the convenience of effective simulation. We can point out under the condition of constant coefficients that CIR-jump model is rather selected as the first estimation method for the more accurate models with variable coefficients. Then we perform HJM-jump model with parameters obtained from previous simulation to obtain more accurate results.

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