# On The Periodic Solutions of Certain Fifth Order Nonlinear Vector Differential Equations

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***Abstract*— The purpose of this paper is to show that under some sufficient conditions of equation (1.1) have no periodic solution other than the trivial solution $\mathbf{X}=\mathbf{0}$.**

***Index Terms*— Nonlinear vector differential equation of fifth, Lyapunov function, Periodic solutions.**

## I. INTRODUCTION

Periodic solutions of high order scalar and vector differential equations have emerged in the applied sciences as some practical mechanical problems, physics, chemistry, biology, ecomomics, control theory. According to observations in the literature, the problems related to the periodic behavior of solutions of a higher order non-linear scalar and vector differential equation have been investigated by many authors. For some related contributors to the subject, we refer to the papers of Li &Duan [1], Bereketoğlu [2-3], Ezeilo [4-6],Tejumola [8],Tiryaki [9] and Tunç[10] . In all of the papers mentioned above, authors used Lyapunov's second (or direct) method [11].

$\dot{x}_i = x_{i+1}\; \dot{x}_i = x_{i+1} \quad (i=1,2,3,4)$

$\dot{x}_5 = -f_5(x_4)x_5 - f_4(x_3)x_4 - f_3(x_1,x_2,x_3,x_4,x_5)x_3$

$-f_2(x_2) - f_1(x_1) \quad (f_2(0) = f_1(0) = 0)$

Produced by Li&Duan [5], we install two new results under different conditions for periodic solution of the solution $X=0$ of the fifth order nonlinear vector differential equations of the form $X^{(5)} + \Psi(\dddot{X})X^{(4)} + \Phi(\ddot{X})\dddot{X} + \Theta(X,\dot{X},\ddot{X},\dddot{X},X^{(4)})\ddot{X}$

$$+E(\dot{X}) + H(X) = 0 \qquad (1.1)$$

in which $X \in \mathbb{R}^n$; $\Psi$, $\Phi$ and $\Theta$ are $n \times n$-symmetric continuous matrix functions depending, in each case ,on the arguments shown; $E, H: \mathbb{R}^n \to \mathbb{R}^n$ are continuous $n$-vector functions. It is supposed that $E(0) = H(0) = 0$.

Let $J_H(X)$, $J_\Psi(W)$, $J_\Phi(Z)$ and $J_E(Y)$ display the Jacobian matrices corresponding to the functions $H(X)$, $\Psi(W)$, $\Phi(Z)$ and $E(Y)$ respectively,

$$J_H(X) = \left(\frac{\partial h_i}{\partial x_j}\right), \quad J_\Psi(W) = \left(\frac{\partial w_i}{\partial w_j}\right)$$

$$J_{\Phi(Z)} = \left(\frac{\partial \Phi_i}{\partial z_j}\right), \; J_E(Y) = \left(\frac{\partial e_i}{\partial y_j}\right), \quad (i,j=1,2,\ldots,n)$$

in which $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$, $(z_1, z_2, \ldots, z_n)$, $(w_1, w_2, \ldots, w_n)$, $(h_1, h_2, \ldots, h_n)$, $(\psi_1, \psi_2, \ldots, \psi_n)$, $(\varphi_1, \varphi_2, \ldots, \varphi_n)$, $(e_1, e_2, \ldots, e_n)$ are the components of $X, Y, Z, W, H, \Psi, \Phi$ and $E$ respectively.The symbol $\langle X, Y\rangle$ corresponding to any pair X,Y in $\Re^n$ stands for he usual scalar product $\sum_{i=1}^{n} x_i y_i$.

Through out this paper, we consider the following differential systems which are equivalent to the equation (1.2) which was attained as usual by setting $\dot{X} = Y, \ddot{X} = Z, \dddot{X} = W, X^{(4)} = U$ from (1.2):

$$\dot{X} = Y, \dot{Y} = Z, \dot{Z} = W, \dot{W} = U$$

$$\dot{U} = -\Psi(W)U - \Phi(Z)W - \Theta(X,Y,Z,W,U)Z - E(Y) - H(X) \quad (1.2)$$

## II. MAİN RESULTS

Our main result is the following two theorems.

**Theorem 2.1.** In addition to the basic conditions given above for coefficients $\Psi, \Phi, \Theta, E$ and $H$ of (1.2) equation, we suppose that following conditions hold as;

(i) $H(0) = 0, H(X) \neq 0$ if $X \neq 0, E(0) = 0, E(Y) \neq 0$ if $Y \neq 0$;

$\Psi, \Phi, \Theta$ symetric and $\lambda_i(J_H(X)) < 0, \quad (i = 1,2,\ldots,n)$,

(ii) $\lambda_i\big(\Theta(X,Y,Z,W,U)\big) \geq 0, \lambda_i(\Psi(W)) < 0$ for all $X,Y,Z,W,U \in \mathbb{R}^n$ $(i = 1,2,\ldots,n)$,

Then equation (1.2) has no periodic solution other than $\mathbf{X} = \mathbf{0}$.

**Proof.** Let

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) = (\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t), \varepsilon_4(t), \varepsilon_5(t))$$

be an arbitrary $\alpha$-periodic solution of (1.2), that is

$$(\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t), \varepsilon_4(t), \varepsilon_5(t))$$
$$= (\varepsilon_1(t+\alpha), \varepsilon_2(t+\alpha), \varepsilon_3(t+\alpha), \varepsilon_4(t+\alpha), \varepsilon_5(t+\alpha)) \quad (2.1)$$

for some $\alpha > 0$.It will be shown that, under the conditions in Theorem (2.1),

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 0.$$

As basic tool the proof of Theorem (2.1), we will use Lyapunov function $V_1 = V_1(X,Y,Z,W,U)$ given as

$$V_1 = -\langle Z,U\rangle - \langle Z,\int_0^1 \langle \Psi(\sigma W),W\rangle d\sigma\rangle + \frac{1}{2}\langle W,W\rangle - \int_0^1 \langle \Phi(\sigma Z)Z,Z\rangle d\sigma$$
$$-\int_0^1 \langle E(\sigma Y),Y\rangle d\sigma - \langle H(X),Y\rangle \qquad (2.2)$$

$\theta(t) = V_1(\varepsilon_1(t),\varepsilon_2(t),\varepsilon_3(t),\varepsilon_4(t),\varepsilon_5(t))$.

Since $\theta(t)$ is continuous and $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5$ are periodic in t, $\theta(t)$ is clearly bounded. An elementary differentiation will show that

$$\dot V_1 = \langle Z,\Psi(W)U\rangle + \langle Z,\Phi(Z)W\rangle + \langle Z,\Theta(X,Y,Z,W,U)Z\rangle$$
$$+\langle Z,E(Y)\rangle - \left\langle \frac{d}{dt}Z,\int_0^1 \langle \Psi(\sigma W),W\rangle d\sigma\right\rangle$$
$$-\frac{d}{dt}\int_0^1 \langle \sigma\Phi(\sigma Z)Z,Z\rangle d\sigma - \frac{d}{dt}\int_0^1 \langle E(\sigma Y),Y\rangle d\sigma - \langle J_H(X)Y,Y\rangle$$

(2.3)

Now ,recall that

$$\frac{d}{dt}\left\langle Z,\int_0^1 \langle \Psi(\sigma W),W\rangle d\sigma\right\rangle + \left\langle Z,\int_0^1 \langle \Psi(\sigma W),U\rangle d\sigma\right\rangle$$
$$+\left\langle Z,\int_0^1 \langle \sigma J_\Psi(\sigma W)U,W\rangle d\sigma\right\rangle$$
$$= \left\langle W,\int_0^1 \langle \Psi(\sigma W),W\rangle d\sigma\right\rangle + \left\langle Z,\int_0^1 \langle \Psi(\sigma W),U\rangle d\sigma\right\rangle$$
$$+\left\langle Z,\int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle \Psi(\sigma W),U\rangle d\sigma\right\rangle$$
$$= \left\langle W,\int_0^1 \langle \Psi(\sigma W),W\rangle d\sigma\right\rangle + \left\langle Z,\int_0^1 \langle \Psi(\sigma W),U\rangle d\sigma\right\rangle$$
$$+\sigma\langle Z,\Psi(\sigma W)U\rangle\Big|_0^1$$
$$= \left\langle W,\int_0^1 \langle \Psi(\sigma W),W\rangle d\sigma\right\rangle + \langle Z,\Psi(W)U\rangle, \qquad (2.4)$$

$$\frac{d}{dt}\int_0^1 \langle \sigma\Phi(\sigma Z)Z,Z\rangle d\sigma = \int_0^1 \langle \sigma\Phi(\sigma Z)W,Z\rangle d\sigma$$
$$+\int_0^1 \langle \sigma\Phi(Z)Z,W\rangle d\sigma + \int_0^1 \sigma^2\langle J_\Phi(Z)ZW,Z\rangle d\sigma$$
$$= \int_0^1 \langle \sigma\Phi(\sigma Z)W,Z\rangle d\sigma + \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle \sigma\Phi(\sigma Z)W,Z\rangle d\sigma$$
$$= \sigma^2\langle \Phi(\sigma Z)W,Z\rangle\Big|_0^1 = \langle \Phi(Z)W,Z\rangle$$

(2.5)

And similary we have

$$\frac{d}{dt}\int_0^1 \langle E(\sigma Y),Y\rangle d\sigma = \int_0^1 \sigma\langle J_E(\sigma Y)Z,Y\rangle d\sigma + \int_0^1 \langle E(\sigma Y),Z\rangle d\sigma$$
$$= \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle E(\sigma Y),Z\rangle d\sigma + \int_0^1 \langle E(\sigma Y),Z\rangle d\sigma$$
$$= \sigma\langle E(\sigma Y),Z\rangle\Big|_0^1 = \langle E(Y),Z\rangle \qquad (2.6)$$

Substituting (2.4), (2.5) and (2.6) into (2.3), and taking into account the conditions of the Theorem (2.1), we obtain

$$\dot V_1 = \frac{d}{dt}V_1(X,Y,Z,W,U) = -\left\langle W,\int_0^1 \langle \Psi(\sigma W),W\rangle d\sigma\right\rangle$$
$$+\langle Z,\theta(X,Y,Z,W,U)Z\rangle - \langle J_H(X)Y,Y\rangle$$

(2.7)

Hence $\dot\theta(t)\ge 0$ ;so that $\theta(t)$ is monotone in *t*, and therefore, being bounded, tends to a limit, $\theta_0$ say, as $t\to\infty$.That is $\lim_{t\to+\infty}\theta(t)=\theta_0$ .it is readily showed that

$\theta(t)=\theta_0$ for all *t*. (2.8)

From by (2.1),

$\theta(t) = \theta(t+m\alpha)$ (2.9)

for any arbitrary fixed *t* an for arbitrary integer *m*, and then letting $m\to\infty$ in the right-hand side of (2.9) leads to (2.8). the result (2.7) itself implies that

$\dot\theta(t)=0$ for all *t*.

Furthermore $\dot\theta(t) = 0$ necessarily implies that $\varepsilon_2=0$ for all $t$.Thus, if $\varepsilon_2=\varepsilon_3=\varepsilon_4=\varepsilon_5=0$ are written in (1.2) system, we can clearly see that it is $H(\varepsilon_1)=H(X_0)=0$, because of $H(0)=0$ and $H(X)\neq 0$ for $X\neq 0$ under conditions of Theorem (2.1) and $\varepsilon_1=X_0$ .So that $\varepsilon_1=X_0=0$ . İt is $\varepsilon_1=\varepsilon_2=\varepsilon_3=\varepsilon_4=\varepsilon_5=0$ for all $t$.Since $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5)$ is a solution of (1.2), it is evident that

$$\big(\varepsilon_1(t),\varepsilon_2(t),\varepsilon_3(t),\varepsilon_4(t),\varepsilon_5(t)\big)=$$
$$\big(\varepsilon_1(t+\alpha),\varepsilon_2(t+\alpha),\varepsilon_3(t+\alpha),\varepsilon_4(t+\alpha),\varepsilon_5(t+\alpha)\big)$$

This completes the proof of Theorem (2.1).

**Theorem 2.2.** In addition to the basic conditions given above for coefficients $\Psi, \Phi, \Theta, E$ and $H$ of (1.2) equation, we suppose that following conditions hold as; (i) $H(0) = 0$, $H(X) \neq 0$ if $X \neq 0$ and $E(0)=0$, $E(Y)\neq 0$ if $Y\neq 0$ as $t \to \infty$

$\Psi$, $\Phi$ ve $\Theta$ symmetric and $\lambda_i$ ( $J_H$ $(X)$)>0, $(i=1,2,...,n.)$,

(ii) $\lambda_i(\Theta(X,Y,Z,W,U)) \leq 0$, $\lambda_i(\Psi(W)) > 0$ for all $X, Y, Z, W, U \epsilon \mathbb{R}^n$,

$\lambda_i(\Phi(Z)) > k$, $k>-1$, $(i=1,2,...,n)$.

Then equation (1.2) has no periodic solution other than $X = 0$

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**Proof.** Let

$$\big(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5\big)=\big(\varepsilon_1(t),\varepsilon_2(t),\varepsilon_3(t),\varepsilon_4(t),\varepsilon_5(t)\big)$$

be an arbitrary $\alpha$-periodic solution of (1.2), that is

$$\big(\varepsilon_1(t),\varepsilon_2(t),\varepsilon_3(t),\varepsilon_4(t),\varepsilon_5(t)\big)=$$
$$\big(\varepsilon_1(t+\alpha),\varepsilon_2(t+\alpha),\varepsilon_3(t+\alpha),\varepsilon_4(t+\alpha),\varepsilon_5(t+\alpha)\big)$$

for some $\alpha > 0$.It will be shown that, under the conditions in Theorem (2.2),

$$\varepsilon_1=\varepsilon_2=\varepsilon_3=\varepsilon_4=\varepsilon_5=0.$$

As basic tool the proof of Theorem (2.1), we will use Lyapunov function $V_2 = V_2(X,Y,Z,W,U)$ given as

$$V_2=\langle Z,U\rangle + < Z,\int_0^1\langle\Psi(\sigma W),W\rangle d\sigma\rangle - \frac{1}{2}\langle W,W\rangle +$$
$$\int_0^1\langle\Phi(\sigma Z)Z,Z\rangle d\sigma + \int_0^1\langle E(\sigma Y),Y\rangle d\sigma + \langle H(X),Y\rangle \quad (2.10)$$

Consider the function

$$\theta(t)=V_2(\varepsilon_1(t),\varepsilon_2(t),\varepsilon_3(t),\varepsilon_4(t),\varepsilon_5(t)).$$

Since $\theta(t)$ is continuous and $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5$ are periodic in t, $\theta(t)$ is clearly bounded. An elementary differentiation will show that

$$\dot{V}_2=-\langle Z,\Psi(W)U\rangle-\langle Z,\Phi(Z)W\rangle-\langle Z,\Theta(X,Y,Z,W,U)Z\rangle$$
$$-\langle Z,E(Y)\rangle+\left\langle\frac{d}{dt}Z,\int_0^1\langle\Psi(\sigma W),W\rangle d\sigma\right\rangle$$
$$+\frac{d}{dt}\int_0^1\langle\sigma\Phi(\sigma Z)Z,Z\rangle d\sigma+\frac{d}{dt}\int_0^1\langle E(\sigma Y),Y\rangle d\sigma+\langle J_H(X)Y,Y\rangle \quad (2.11)$$

Substituting (2.4) ,(2.5) and (2.6) into (2.11) and taking into account the conditions of the Theorem (2.2), we obtain

$$\dot{V}_2=\left\langle W,\int_0^1\langle\Psi(\sigma W),W\rangle d\sigma\right\rangle-\langle Z,\Theta(X,Y,Z,W,U)Z\rangle$$
$$+\langle J_H(X)Y,Y\rangle>0 \quad (2.12)$$

Hence $\dot{\theta}(t)\geq 0$ ;so that $\theta(t)$ is monotone in $t$, and therefore, being bounded, tends to a limit, $\theta_0$ say, as $t\to\infty$.That is $\lim_{t\to+\infty}\theta(t)=\theta_0$ .it is readily showed that $\theta(t)=\theta_0$ for all $t$. (2.13)

From by (2.1),

$$\theta(t) = \theta(t+m\alpha) \quad (2.14)$$

for any arbitrary fixed $t$ an for arbitrary integer $m$, and then letting $m\to\infty$ in the right-hand side of (2.14) leads to (2.13). the result (2.12) itself implies that

$$\dot{\theta}(t)=0 \text{ for all } t.$$

Furthermore $\dot{\theta}(t) = 0$ necessarily implies that $\varepsilon_2$=0 for all $t$.Thus, if $\varepsilon_2=\varepsilon_3=\varepsilon_4=\varepsilon_5=0$ are written in (1.2) system, we can clearly see that it is $H(\varepsilon_1)=H(\eta)=0$, because of $H(0)=0$ and $H(X)\neq 0$ for $X\neq 0$ under conditions of Theorem (2.2) and $\varepsilon_1=\eta$ .So that $\varepsilon_1=\eta=0$ .İt is $\varepsilon_1=\varepsilon_2=\varepsilon_3=\varepsilon_4=\varepsilon_5=0$ for all $t$.Since $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5)$ is a solution of (1.2), it is evident that

$$\big(\varepsilon_1(t),\varepsilon_2(t),\varepsilon_3(t),\varepsilon_4(t),\varepsilon_5(t)\big)=$$
$$\big(X(t),Y(t),Z(t),W(t),U(t)\big)=(0,0,0,0,0)$$

This completes the proof of Theorem (2.2).

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