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Integral solutions of the heptic equation with five unknowns $(x^2-y^2)[c^2(x^2+y^2)-2(c^2-1)xy]=(2c^2+3)[X^2-Y^2]z^5$

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***Abstract*— The non-homogeneous Diophantine equation of degree seven with five variables represented by $(x^2-y^2)[c^2(x^2+y^2)-2(c^2-1)xy]=(2c^2+3)[X^2-Y^2]z^5$ is analyzed for its non-zero distinct integer solutions. A few interesting relation between the solutions and special numbers namely Polygonal numbers, Pyramidal numbers, centered Polygonal numbers are exhibited.**

***Index Terms*— heptic, Integral solutions, non-homogeneous equation. M.sc 2000 mathematics subject classification: 11D41**

I. Introduction

Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity as can be seen from [1,4]. The problem of finding all integer solutions of a diophantine equation with three or more variables and degree at least three, in general presents a good deal of difficulties. There is vast general theory of homogeneous quadratic equations with three variables. Cubic equations in two variables fall into the theory of elliptic curves which is a very developed theory but still an important topic of current research. A lot is known about equations in two variables in higher degrees. Very few equations with more than three variables and degree at least three is known. It is worth to note that undesirability appears in equations, even perhaps at degree four with fairly small co-efficient. It seems that much work has not been done in solving higher order Diophantine equations. In [5-26] a few higher order equations are considered for integral solutions. In this communication a seventh degree non-homogeneous equation, with five variables represented by $(x^2-y^2)[c^2(x^2+y^2)-2(c^2-1)xy]=(2c^2+3)[X^2-Y^2]z^5$ is considered and a few interesting relations among the solutions are presented.

II NOTATIONS

$t_{m,n}$: Polygonal number of rank n with size m

P_n^m : Pyramidal number of rank n with size m

CP_n^m : Centered Pyramidal number of rank n with size m.

J_n :Jacobsthal number of rank n

j_n : Jacobsthal-Lucas number of rank n

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III Method of analysis

The non-homogeneous heptic equation with five unknowns to be solved for its distinct non-zero integral solutions is

$$(x^2-y^2)[c^2(x^2+y^2)-2(c^2-1)xy]=(2c^2+3)[X^2-Y^2]z^5 \qquad (1)$$

Introduction of the linear transformations

$$x=au+bv, \quad y=au-bv, \quad X=2au+bv, \quad Y=2au-bv \qquad (2)$$

in (1) leads to

$$a^2u^2+b^2v^2(2c^2-1)=(2c^3+3)z^5 \qquad (3)$$

Different methods of obtaining the patterns of integer solutions to (1) are illustrated below:

Pattern:1

Let

$$z=a^2b^2[p^2+(2c^2-1)q^2] \qquad (4)$$

Write $(2c^3+3)$ as

$$(2c^3+3)=[2+i\sqrt{(2c^2-1)}][2-i\sqrt{(2c^2-1)}] \qquad (5)$$

Using (4), (5) in (3) and applying the method of factorization, define

$$\left(au+ibv\sqrt{2c^2-1}\right)=\left(2+i\sqrt{2c^2-1}\right)[a^5b^5\left(\alpha+i\beta\sqrt{2c^2-1}\right)] \qquad (6)$$

where

$$\left(\alpha+i\beta\sqrt{2c^2-1}\right)=[p+iq\sqrt{2c^2-1}]^5$$

from which we have

$$\left.\begin{aligned}\alpha&=p^5-10p^3(2c^2-1)q^2+5p(2c^2-1)^2q^4\\ \beta&=5p^4q-10p^2(2c^2-1)q^3+(2c^2-1)^2q^5\end{aligned}\right\} \qquad (7)$$

Equating real and imaginary parts in (6), we get

$$\left.\begin{aligned}au&=a^5b^5[2\alpha-(2c^2-1)\beta]\\ bv&=a^5b^5[\alpha+2\beta]\end{aligned}\right\} \qquad (8)$$

Using (8) and (2), the value of x,y,z, X and Y are given by,

$$\left.\begin{aligned}x(a,b)&=a^5b^5[3\alpha+(3-2c^2)\beta]\\ y(a,b)&=a^5b^5[\alpha-(1+2c^2)\beta]\\ X(a,b)&=a^5b^5[5\alpha+4(1-c^2)\beta]\\ Y(a,b)&=a^5b^5[3\alpha-(4c^2)\beta]\end{aligned}\right\} \qquad (9)$$

Thus, (4) and (9) represent the non-zero distinct integral solutions to (1)

Properties:

$(i)x(a,b,c,n,1)-Y(a,b,c,n,1)=a^5b^5(3+2c^2)[5t_{4,n}(t_{4,n}+2)-20c^2t_{4,n}+(2c^2-1)^2]$

$(ii)x(a,b,c,n,1)-X(a,b,c,n,1)+Y(a,b,c,n,1)-y(a,b,c,n,1)$
$\qquad =-2a^5b^5[24F_{4,n,7}-6CP_n^{13}-2CP_n^3-8t_{3,n}+t_{4,n}-20c^2t_{4,n}+12(c^2-1)^2]$

$(iii)z(a,b,c,2^n,2^{n+1})=a^2b^2[j_{2n}+3(2c^2-1)J_{2n+2}+(2c^2-2)]$

Note:

In (2), the transformations for X and Y may be taken as

$$X=2auv+b, \quad Y=2auv-b \qquad (10)$$

for which we have

$$X(a,b,c,p,q)=2a^{10}b^9[2\alpha^2+\alpha\beta(5-2c^2)+\beta^2(2-4c^2)]+b$$

$$Y(a,b,c,p,q)=2a^{10}b^{9}\begin{bmatrix}\alpha^2(1+c^2)+\alpha\beta(-2c^6+c^4+4c^2-2)+\\ \beta^2(-2c^4+3c^2+1)\end{bmatrix}-b$$

Pattern:2

Instead of (5), we write $(2c^3+3)$ as

$$(2c^3+3)=\frac{[1+i(c^2+1)\sqrt{2c^2-1}][1-i(c^2+1)\sqrt{2c^2-1}]}{c^4}\quad(11)$$

Following the procedure similar to the above, the corresponding integer solutions to (1) are obtained as

$x(a,b,c,p,q)=a^5b^5c^3[\alpha(c^2+2)-\beta(2c^4+c^2-2)]$
$y(a,b,c,p,q)=a^5b^5c^3[-\alpha c^2-\beta(2c^4+c^2)]$
$X(a,b,c,p,q)=a^5b^5c^3[\alpha(c^2+3)-\beta(4c^4+2c^2-3)]$
$Y(a,b,c,p,q)=a^5b^5c^3[\alpha(1-c^2)-\beta(4c^4+2c^2-1)]$

Properties:

$*\ (2c^4+c^2)x(a,b,c,1,n)-(2c^4+c^2-1)y(a,b,c,1,n)$
$=a^5b^5(4c^9+6c^7)\{1-\quad 10(2c^2-1)t_{4,n}+(2c^2-1)^2[24F_{4,n,7}-6CP_n^{13}-2CP_n^3-2t_{9,n}+2t_{3,n}-t_{4,n}]\}$

$(ii)(1-c^2)X(a,b,c,n,1)-(c^2+3)Y(a,b,c,n,1)$
$=a^5b^5(-16c^7-10c^5+6c^2)[24F_{4,n,7}-6CP_n^{13}-2CP_n^3-t_{16,n}-20t_{3,n}$
$+10t_{4,n}-(2C^2-1)10t_{4,n}+(2c^2-1)^2]$

$z(a,b,c,2^{n+1},2^n)=a^2b^2c^2[KY_{n+1}+(2c^2-1)(3J_{2n}+1)-(3J_{n+2}+(-1)^{n+2})+1]$

Note:

For the choice of X,Y given by (10), we have

$$X(a,b,c,p,q)=2\,a^{10}b^9c^6\begin{bmatrix}\alpha^2(1+c^2)+\alpha\beta(-2c^6+c^4+4c^2-2)\\ +\beta^2(-2c^4+3c^2+1)\end{bmatrix}+b$$

$$Y(a,b,c,p,q)=2\,a^{10}b^9c^6\begin{bmatrix}\alpha^2(1+c^2)+\alpha\beta(-2c^6+c^4+4c^2-2)\\ +\beta^2(-2c^4+3c^2+1)\end{bmatrix}-b$$

Pattern:3

Consider (3) as

$$a^2u^2+b^2v^2(2c^2-1)=(2c^3+3)z^5*1\qquad(12)$$

Write 1 as

$$1=\frac{[(c^2-1)+i\sqrt{2c^2-1}][(c^2-1)-i\sqrt{2c^2-1}]}{c^4}\qquad(13)$$

substituting (4),(5) and (13) in (12) and employing the factorization method, define

$$\left(au+ibv\sqrt{2c^2-1}\right)$$
$$=(2+i\sqrt{2c^2-1})a^5b^5(p+iq\sqrt{2c^2-1})^5\,\frac{(c^2-1)+i\sqrt{2c^2-1}}{c^2}$$

Equating real and imaginary parts we have

$$\left.\begin{aligned}au&=\frac{a^5b^5}{c^2}[(c^2-1)[2\alpha-(2c^2-1)\beta]-(2c^2-1)[\alpha+2\beta]]\\ bv&=\frac{a^5b^5}{c^2}[(2\alpha-(2c^2-1)\beta)+(c^2-1)(\alpha+2\beta)]\end{aligned}\right\}\qquad(14)$$

As our interest is on finding integer solutions we choose p and q suitably so that au and bv are integers. Replace p by cp and q by cq in (7), substituting the corresponding values of α and β in (14) and employing (2) the non-zero integral solutions to (1) are given by

$x(a,b,c,p,q)=a^5b^5c^3[\alpha c^2-\beta(2c^4+c^2)]$
$y(a,b,c,p,q)=a^5b^5c^3[\alpha(-c^2-2)-\beta(2c^4+c^2-2)]$
$X(a,b,c,p,q)=a^5b^5c^3[\alpha(c^2-1)-\beta(4c^4+2c^2-1)]$
$Y(a,b,c,p,q)=a^5b^5c^3[\alpha(-c-3)-\beta(4c^4+2c^2\qquad-3)]$
$z(a,b,c,p,q)=a^2b^2c^2[p^2+(2c^2-1)q^2)]$

Properties:

$(i)(c^2-1)x(a,b,c,n,1)-c^2X(a,b,c,n,1)$
$=a^5b^5c^3(2c^6+3c^4)[F_{4,n,7}-6CP_n^{17}-2CP_n^9-CP_{14,n-}2t_{3,n}+t_{4,n}-1$
$-10t_{4,n}(2c^2-1)+(2c^2-1)^2]$

$(ii)(c^2+3)y(a,b,c,n,1)+(c^2-1)Y(a,b,c,n,1)$
$=a^5b^5c^3(-6c^6-c^4+6c^2)[F_{4,n,7}-6CP_n^{13}+CP_n^6-CP_{10,n}-2t_{4,n}-1$
$-10t_{4,n}(2c^2-1)+(2c^2-1)^2]$

$(iii)z(a,b,c,2^{n+1},2^{n+1})=a^2b^2c^2[3J_{2n+2}+1-(2c^2-1)(j_{2n+2}-1)]$
$(iii)z(a,b,c,2^{n+1},2^{n+1})$
$=a^2b^2c^2[3J_{2n+2}+1-(2c^2-1)(j_{2n+2}-1)]$

Note:

Substituting (10) in (2), we get

$X(a,b,c,p,q)=2a^{10}b^9c^6[-\alpha(c^2+1)+\alpha\beta(-6c^5-6c^3+3c^2+4)$
$+\beta^2(6c^2-3)]+b$

$Y(a,b,c,p,q)=2a^{10}b^9c^6[-\alpha(c^2+1)$
$+\alpha\beta(-6c^5-6c^3+3c^2+4)+\beta^2(6c^2-3)]-b$

Pattern:4

Instead of using (5), we use (11) in pattern.3. The non-zero distinct integral solutions to (1) are found to be

$x(a,b,c,p,q)=a^5b^5c[\alpha c^4-\beta(2c^6+c^4)]$
$y(a,b,c,p,q)=a^5b^5c[-3\alpha c^4-\beta(2c^6-3c^4+2c^2)]$
$X(a,b,c,p,q)=a^5b^5c[-3\alpha c^4-4\beta c^6]$
$Y(a,b,c,p,q)=a^5b^5c[-5\alpha c^4-\beta(4c^6-4c^4)]$

Properties:

$(i)3x(a,b,c,n,1)-X(a,b,c,n,1)=a^5b^5c(-2c^7-3c^5)$
$[F_{4,n,7}-6CP_n^{13}-2CP_n^3-CP_{8,n}-1-3t_{4,n}-10t_{4,n}(2c^2-1)+(2c^2-1)^2]$
$(ii)5X(a,b,c,n,1)-3Y(a,b,c,n,1)=a^5b^5c(12c^4-8c^6)$
$[F_{4,n,7}-6CP_n^{14}-t_{16,n}-10t_{4,n}(2c^2-1)(2c^2-1)^2]$
$(iii)z(a,b,c,2^n,2^n)=a^2b^2c^2[3J_{2n}+1+(2c^2-1)(j_{2n}-1)]$

Note:

For the choice of X and Y given by (10) we have

$X(a,b,c,p,q)=2a^{10}b^9c^2[-\alpha^2(4+2c^2)+\alpha\beta(-2c^6-3c^4-4c^2+6)$
$-\beta^2(2-3c^2-2c^4)(2c^4+1)+\beta(2-3c^2-2c^4)-2\alpha]+b$
$Y(a,b,c,p,q)=2a^{10}b^9c^2[-\alpha^2(4+2c^2)+\alpha\beta(-2c^6-3c^4-4c^2+6)$
$-\beta^2(2-3c^2-2c^4)(2c^4+1)+\beta(2-3c^2-2c^4)-2\alpha]-b$

CONCLUSION

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the non-homogeneous heptic equation with five variables. As the heptic equations are rich in variety, one may search for other forms of heptic equation with variables greater than or equal to five and obtain their corresponding properties.

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