Multiobjective Non-Differentiable Fractional Symmetric Mixed Duality using ρ-F convexity

Gayatri Devi, Rashmita Jena

Abstract— **Mixed symmetric dual models far nondifferentiable multiobjective fractional programmeing problem are introduced. Weak and strong duality theorems are established far these models under generalized convexity. Several special cases are also obtained.**

Index Terms— **Non differentiable fractional programming, symmetric duality, generalized convexity, ρ-function.**

I. INTRODUCTION

Multiobjective fractional programming duality has been of much interest in the recent past. Schaible [8] and Bectar et al [2] derived Fritz John and Karush-Kuhn Tucker necessary and sufficient optimality condition for a class of nondifferentiable convex multiobjective fractional programming problems and established some duality theorems. Liang et al [3] and santos et al [7] discussed the optimality and duality for nonsmooth fractional programming with generalized convexity. Bectar et al [1] and Xu [9] gave a mixed type duality for fractional programming, established some sufficient condition and obtained various duality reseults between the mixed dual and primal problem.

Several authors, such as the ones of [4, 5, 6, 10, 11], studied second and higher order symmetric duality.

In this paper we introduced new type of mixed symmetric dual models for non-differentiable multiobjective fractional programming problem. The advantage of this model is that it allows further weakning of convexity on the functions invalued. We establish weak and strong duality theorems for this model.

II. PRELIMINARIES :

Let C be a compact set in R^n. The support function of C is defined by

$$s(x\,|\,c) = \max\left\{x^T y : y \in c\right\}.$$

Definition : 1

Let $X \subset R^n$. A functional $F : X \times X \times R^n \to R$ is said to be sublinear with respect to its third argument if for any x, y ∈ X

(i) $F(x,y;a_1+a_2) \le F(x,y;a_1) + F(x,y;a_2)$ for any $a_1, a_2 \in R^n$;

(ii) $F(x,y;\alpha a) \le \alpha F(x,y;a)$ for any $\alpha \in R_+$ and $a \in R^n$

Gayatri Devi, Rashmita Jena, Prof. in : CSE, ABIT College, CDA-1, Cuttack

Definition : 2

A differentiable function f is said to be ρ-F convex at $\bar{x} \in y$ if

$$f(x,y) - f(\bar{x},y) \ge F\left(x,\bar{x},\nabla_x f(\bar{x},y)\right) + \rho\left\|x-\bar{x}\right\|^2, \forall x \in X$$

Definition : 3

f is said to be ρ – F pseudoconvex at $\bar{x} \in X$ if

$$F\left(x,\bar{x}_i,\nabla_x f(\bar{x},y)\right) + \rho\left\|x-\bar{x}\right\|^2 \ge 0$$

$$\Rightarrow f(x,y) \ge f(\bar{x},y)$$

Definition : 4

f is said to be ρ-F quasiconvex at $\bar{x} \in X$

if $f(x,y) - f(\bar{x},y) \le 0$

$$\Rightarrow F\left(x,\bar{x};\nabla_x f(\bar{x},y)\right) + \rho\left\|x-\bar{x}\right\|^2 \le 0 \quad \forall x \in X$$

Let $x = (x^1, x^2),\ x^1 \in R^{|J_1|},\ x^2 \in R^{|J_2|}$

and $f : R^{|J_1|} \times R^{|K_1|} \to R^{\ell}$

Mixed type multiobjective fractional dual
Primal Problem

$$(\text{MFP}): \text{minimize}\frac{f(x,y)}{g(x,y)} = \frac{f_{1i}(x^1y^1)}{g_{1i}(x^1y^1)} + \frac{f_{2i}(x^2,y^2)}{g_{2i}(x^2,y^2)}$$

$$= \left(f_{1i}(x^1\ y^1) - \nu_{1i}g_{1i}(x^1y^1)\right) + \left(f_{2i}(x^2,y^2) - \nu_{2i}g_{2i}(x^2,y^2)\right)$$

$$= f_{1i}(x^1,y^1) - \nu_{1i}g_{1i}(x^1,y^1) + s(x^1\,|\,C_i^1) + f_{2i}(x^2,y^2) - \nu_{2i}g_{2i}(x^2,y^2)$$

$$s(x^2\,|\,C_i^2) - (y^1)^T z_i^1 - (y^2)^T z_i^2 \text{ for } i = 1,\ldots.. \ \ell,$$

subject to

$$(x^1,x^2,y^1,y^2,z^1,z^2,\lambda) \in R^{|J_1|} \times R^{|J_2|} \times R^{|K_1|} \times R^{|K_2|},$$

$$\sum_{i=1}^{l}\lambda_i\left[\nabla_{y^1}\left(f_{1i}(x^1,y^1) - \nu_{1i}g_{1i}(x^1,y^1)\right) - z_i^1\right] \le 0,$$

$$\sum_{i=1}^{l}\lambda_i\left[\nabla_{y^2}\left(\rho_{2i}(x^2,y^2) - \nu_{2i}g_{2i}(x^2,y^2)\right) - z_i^2\right] \le 0,$$

$$\left(y^1\right)^T \sum_{i=1}^{l} \lambda_i \left[\nabla_{y^1}\left(f_{1i}\left(x^1,y^1\right)-\nu_{1i}g_{1i}\left(x^1,y^1\right)\right)-z_i^1\right]\geq 0$$

$$\left(y^2\right)^T \sum_{i=1}^{l} \lambda_i \left[\nabla_{y^2}\left(f_{2i}\left(x^2,y^2\right)-\nu_{2i}g_{2i}\left(x^2,y^2\right)\right)-z_i^2\right]\geq 0$$

$\left(x^1,x^2\right)\geq 0,\ z_i^1 \in D_i^1$ and $z_i^2 \in D_i^2,\ i=1,2,......,\ell$

Dual Problem (MFD)

maximize

$$\frac{f(u,v)}{g(u,v)}=\frac{f_{1i}\left(u^1,v^1\right)}{g_{1i}\left(u^1,v^1\right)}+\frac{f_{2i}\left(u^2,v^2\right)}{g_{2i}\left(u^2,v^2\right)},i=1,.....,\ell$$

$$=f_{1i}\left(u^1,v^1\right)-\nu_{1i}g_{1i}\left(u^1,v^1\right)$$

$$+f_{2i}\left(u^2,v^2\right)-\nu_{2i}g_{2i}\left(u^2,v^2\right)$$

$$=f_{1i}\left(u^1,v^1\right)-\nu_{1i}g_{1i}\left(u^1,v^1\right)+f_{2i}\left(u^2,v^2\right)-\nu_{2i}g_{2i}\left(u^2,v^2\right)$$

$$-\ s\left(v^1\,|\,D_i^1\right)-s\left(v^2\,|\,D_i^2\right)-$$

$$\left(u^1\right)^T\omega_i^1+\left(u^2\right)^T\omega_i^2$$

subject to

$$\left(u^1,u^2,v^1,v^2,\omega^1,\omega^2,\lambda\right)\in R^{|J_1|}\times R^{|J_2|}\times R^{|K_1|}\times R^{|K_2|}\times R^{|k_1|}\times R^{|k_2|}\times R_+^{\ell}$$

$$\sum_{i=1}^{l}\lambda_i\left[\nabla_{x^1}\left(f_{1i}\left(u^1,v^1\right)-\nu_{1i}g_{1i}\left(u^1,v^1\right)\right)+\omega_i^1\right]\geq 0,$$

$$\sum_{i=1}^{l}\lambda_i\left[\nabla_{x^2}\left(f_{2i}\left(u^2,v^2\right)-\nu_{2i}g_{2i}\left(u^2,v^2\right)\right)+\omega_i^2\right]\geq 0,$$

$$\left(u^1\right)^T\sum_{i=1}^{l}\lambda_i\left[\nabla_{x^1}\left(f_{1i}\left(u^1,v^1\right)-\nu_{1i}g_{1i}\left(u^1,v^1\right)\right)+\omega_i^1\right]\leq 0,$$

$$\left(u^2\right)^T\sum_{i=1}^{l}\lambda_i\left[\nabla_{x^2}\left(f_{2i}\left(u^2,v^2\right)-\nu_{2i}g_{2i}\left(u^2,v^2\right)\right)+\omega_i^2\right]\leq 0,$$

$$\left(v^1,v^2\right)\geq 0,$$

$$\omega_i^1\in C_i^1 \text{ and } \omega_i^2\in C_i^2,\ i=1,2...\ell.$$

$$\lambda>0,\ \sum_{i=1}^{l}\lambda_i=1$$

C_i^1 is a compact and convex subsets of $R^{|J_1|}$ and C_i^2 is a compact and convex subsets of $R^{|J_2|}$ for i = 1, ... ℓ

Similarly D_i^1 is a compact and convex subsets of $R^{|K_1|}$ and D_i^2 is a compact and convex subsets of $R^{|K_2|}$ for i = 1, ... ℓ

Theorem - 1 (Weak duality)

Let $\left(x^1,x^2,y^1,y^2,z^1,z^2,\lambda\right)$ be feasible for (MFP) and $\left(u^1,u^2,v^1,v^2,w^1,w^2,\lambda\right)$ be feasible for (MFD). Suppose for i = 1,.., ℓ

$\left(f_{1i}\left(.,\,y^1\right)-\nu_{1i}g_{1i}\left(.,\,y^1\right)\right)+.^T w_i^1$ is $\rho-F_1$ convex for fixed y^1,

$\left(f_{1i}\left(x^1,.\right)-\nu_{1i}g_{1i}\left(x^1.\right)\right)-.^T z_i^1$ is $\rho-F_2$ concave for fixed x^1,

$\left(f_{2i}\left(.,y^2\right)-\nu_{2i}g_{2i}\left(.,\,y^2\right)\right)+.^T w_i^2$ is $\rho-G_1$ convex for fixed y^2 and

$\left(f_{2i}\left(x^2.^T\right)-\nu_{2i}\ g_{2i}\left(x^2,.\right)\right)-.^T\ z_i^2$ is $\rho-G_2$ concave for fixed x^2 and the following conditions are satisfied.

(i) $F_1\left(x^1,u^1;a\right)+\left(u^1\right)^T a\geq 0,\ \text{if}\ a\geq 0$

(ii) $G_1\left(x^2,u^2;b\right)+\left(u^2\right)^T b\geq 0,\ \text{if}\ b\geq 0$

(iii) $F_2\left(y^1,v^1;c\right)+\left(y^1\right)^T c\leq 0,\ \text{if}\ c\leq 0$ and

(iv) $G_2\left(y^2,v^2;d\right)+\left(y^2\right)^T d\leq 0,\ \text{if}\ d\leq 0$.

Then $\dfrac{f(x,y)}{g(x,y)}\not\leq\dfrac{f(u,v)}{g(u,v)}$.

Proof.

Suppose $\left(x^1,x^2,y^1,y^2,z^1,z^2,\lambda\right)$ be feasible for (MFP) and $\left(u^1,u^2,v^1,v^2,w^1,w^2,\lambda\right)$ be feasible for (MFD)

By using $\rho-F_1$ convexity of $\left(f_{1i}\left(.,\,y^1\right)-\nu_{1i}g_{1i}\left(.,\,y^1\right)\right)+.^T\ w_i^1$ and ρ-F_2 concavity of $\left(f_{1i}\left(x^1,.\right)-\nu_{1i}g_{1i}\left(.,\,y^1\right)\right)+.^T z_i^1$ for i = 1, .. ℓ,

we get

$$\left(f_{1i}\left(x^1,v^1\right)-\nu_{1i}g_{1i}\left(x^1,\,v^1\right)\right)+\left(x^1\right)^T w_i^1 -$$

$$\left(f_{1i}\left(u^1,v^1\right)-\nu_{1i}g_{1i}\left(u^1,\,v^1\right)\right)-\left(u^1\right)^T w_i^1\geq$$

$$F_1\left(x^1,u^1;\nabla_{x^1}\left(f_{1i}\left(u^1,\,v^1\right)-\nu_{1i}g_{1i}\left(u^1,\,v^1\right)+w_i^1\right)\right.$$

$$+\rho\left\|x^1-u^1\right\|^2$$

$$\geq F_1\left(x^1,u^1,\nabla_{x^1}\left(f_{1i}\left(u^1,v^1\right)-\nu_{1i}g_{1i}\right)\left(u^1,v^1\right)+w_i^1\right)$$

a

$$\left(f_{1i}\left(x^1,v^1\right)-\nu_{1i}g_{1i}\left(x^1,v^1\right)\right)-\left(v^1\right)^T z_i^1-$$

$$\left(f_{1i}\left(x^1,y^1\right)-\nu_{1i}g_{1i}\left(x^1,y^1\right)\right)+\left(y^1\right)^T z_i^1\leq$$

$$F_2\left(v^1,y^1;\nabla_{y^1}\left(f_{1i}\left(x^1,y^1\right)-\nu_{1i}g_{1i}\left(x^1,y^1\right)\right)-z_i^1\right)+$$

$$\rho\left\|v^1-y^1\right\|\leq F_2\left(v^1,y^1;\nabla_{y^1}f_{1i}\left(x^1,y^1\right)-\nu_{1i}g_{1i}\left(x^1,y^1\right)-z_i^1\right)$$

From above equations and the sublinearity of F_1 and F_2 we get

$$\sum_{i=1}^{l}\lambda_i\Big[\left(f_{1i}\left(x^1,v^1\right)-\nu_{1i}g_{1i}\left(x^1,v^1\right)\right)+\left(x^1\right)^T w_i^1$$
$$-\left(f_{1i}\left(u^1,v^1\right)-\nu_{1i}g_{1i}\left(u^1,v^1\right)\right)-\left(u^1\right)^1 w_i^1\Big]\geq$$

$$F_1\left(x^1,u^1,\nabla_{x^1}\left(f_{1i}\left(u^1,v^1\right)-\nu_{1i}g_{1i}\right)\left(u^1,v^1\right)\right)+w_i^1$$

and

$$\sum_{i=1}^{l}\lambda_i\Big[\left(f_{1i}\left(x^1,v^1\right)-\nu_{1i}g_{1i}\left(x^1,v^1\right)\right)-\left(v^1\right)^T z_i^1$$
$$f_{1i}\left(x^1,y^1-\nu_{1i}g_{1i}\left(x^1,y^1\right)\right)+\left(y^1\right)^T z_i^1\Big]$$

$$\leq F_2\left(v^1,y^1;\sum_{i=1}^{l}\lambda_i\left[\nabla_{y^1}\left(f_{1i}\left(x^1,y^1\right)-\nu_{1i}g_{1i}\left(x^1,y^1\right)\right)-z_i^1\right]\right)$$

From constraints conditions, we get

$$\sum_{i=1}^{l}\lambda_i\Big[\left(f_{1i}\left(x^1,v^1\right)-\nu_{1i}g_{1i}\left(x^1,v^1\right)+\left(x^1\right)^T w_i^1\right)-$$
$$\left(f_{1i}\left(u^1,v^1\right)-\nu_{1i},g_{1i}\left(u^1,v^1\right)\right)-\left(u^1\right)^T w_i^1\Big]\geq 0,$$

$$\sum_{i=1}^{l}\lambda_i\Big[\left(f_{1i}\left(x^1,v^1\right)-\nu_{1i}g_{1i}\left(x^1,v^1\right)-\left(v^1\right)^T z_i^1\right)-$$
$$\left(f_{1i}\left(x^1,y^1\right)-\nu_{1i},g_{1i}\left(x^1,y^1\right)\right)+\left(y^1\right)^T z_i^1\Big]\leq 0.$$

Rearranging, we obtain

$$\sum_{i=1}^{l}\lambda_i\Big[\left(f_{1i}\left(x^1,y^1\right)-\nu_{1i}g_{1i}\left(x^1,y^1\right)\right)-$$
$$\left(f_{1i}\left(u^1,v^1\right)-\nu_{1i}g_{1i}\left(u^1,v^1\right)\right)+\left(x^1\right)^T w_i^1-\left(u^1\right)^T w_i^1$$
$$+\left(v^1\right)^T z_i^1-\left(y^1\right)^T z_i^1\Big]\geq 0$$

Using

$\left(v^1\right)^T z_i^1\leq s\left(v^1\,|\,D_i^1\right)$ and $\left(x^1\right)^T w_i^1\leq s\left(x^1\,|\,C_i^1\right)$ for i = 1, ... ℓ

We have

$$\sum_{i=1}^{\ell}\lambda_i\Big[\left(f_{1i}\left(x^1,y^1\right)-\nu_{1i}g_{1i}\left(x^1,y^1\right)\right)-$$
$$\left(f_{1i}\left(u^1,v^1\right)-\nu_{1i}g_{1i}\left(u^1,v^1\right)\right)+s\left(x^1\,|\,C_i^1\right)-\left(u^1\right)^T w_i^1$$
$$+s\left(v^1\,|\,D_i^1\right)-\left(y^1\right)^T z_i^1\Big]\geq 0$$

similarly,

$$\sum_{i=1}^{\ell}\lambda_i\Big[\left(f_{2i}\left(x^2,y^2\right)-\nu_{2i}g_{2i}\left(x^2,y^2\right)\right)+s\left(x^2\,|\,C_i^2\right)-\left(y^2\right)^T z_i^2-$$

$$\left(f_{2i}\left(u^2,v^2\right)-\nu_{2i}g_{2i}\left(u^2,v^2\right)\right)+s\left(v^2\,|\,D_i^2\right)-\left(u^2\right)^T w_i^2\Big]\geq 0.$$

$$\Rightarrow\frac{f\left(x,y\right)}{g\left(x,y\right)}\not\leq\frac{f\left(u,v\right)}{g\left(u,v\right)}.$$

Strong Duality Theorem :

Let $\left(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{z}^1,\bar{z}^2,\lambda\right)$ be an efficient solution for (MFP) such that the Hessian matrices $\nabla_{x^1}^2\lambda^T\frac{f\left(\bar{x}^1,\bar{y}^1\right)}{g\left(\bar{x}^1,\bar{y}^1\right)}$ and $\nabla_{y^2}^2\lambda^T\frac{f\left(\bar{x}^2,\bar{y}^2\right)}{g\left(\bar{x}^2,\bar{y}^2\right)}$ be positive definite. Also the sets $\left\{\nabla_{y^1}\frac{f_{1i}}{g_{1i}}-z_i^1\right\},\left\{\nabla_{y^2}\frac{f_{2i}}{g_{2i}}-z_i^2\right\}$ for i = 1,.....*l* linearly independent. If the ρ-F convexity hypothesis and conditions of theorems 3.1 are satisfied, then $\left(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{z}^1,\bar{z}^2,\lambda\right)$ is an efficient solution for (MFD).

III. Special cases

In this section, we consider some special cases of problems (MFP) and (MFD).

(i) If $l = 1$ then (MFP) and (MFD) reduce to the pair of single objective fractional programming problem.

(ii) These results can be further extended to second order mixed type duality.

(iii) Also the present work can be further extended to a class of minimax mixed fractional problems.

REFERENCES

[1] Bector, C.R., Chandra, S. and Abha (2001) : On mixed symmetric duality in Mathematical Programming, J.Math, Anal. Appl. 259, 346-356.

[2] Bector, C.R., Chandra, s. and Husain, I.(1993); Optimality conditions and subdifferentiable multiobjective fractional programming, J.Optim. Theory Appl. 79, 100-125.

[3] Liang, Z., Huang, H.X. and Pardalos, P.M. (2003) : Efficiency condition and duality for a class of multiobjective fractional programming problem, J. Global Optim, 27, 447-471.

[4] Mishra, S.K., Wang, S.Y. and Lai, K.K; (2006) : Mond-Weir type mixed symmetric first and second order duality in nondifferentiable mathematical programming, J. Nonlinear convex Anal., 7(3), 189-198.

[5] Mishra, S.K, Wang, S.Y. and Lai K.K. and Yang, F.M. (2007) : Mixed symmetric duality in nondifferentiable mathematical programming, European J. Oper. Res. 181, 1-9.

[6] Mukherjee, R.N (1991) : Generalized convex duality for multiobjective fractional programs, J. Mahi Anal. Appl., 162, 309-316.

[7] Santos, L.B. et al (2008) : Non-smooth multiobjective fractional programming with generalized convexity, Rev. Integs. Temas. Mat., 26(1), 1-12.

[8] Schaible, S. (1995) : Fractional Programming, Handbook of Global Optimization, Edited by Horst, R and Pardalos, P.M., Kluwer Academic Publishers, Netherland 495-608.

[9] Xu. Z. (1996) : Mixed type duality in multiobjective Programming problem, J. Math. Anal., 198, 621-635.

[10] Mond, (1974) : A class of nondifferentiable mathematical programming problems, J. Math, Anal. Appl. 46, 169-174.

[11] Mond, B (1978) : A class of nondifferentiable fractional programming, Z. Angew. Math. Mech, 58, 337-341.