A multi attribute decision making method based on inclusion measure for interval neutrosophic sets

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Abstract— **In this paper, we introduce an inclusion measure for interval neutrosophic sets, which is one of information measures of interval neutrosophic theory. For this purpose, we first give a system of axioms for inclusion measure of interval neutrosophic sets. Using the concept of distance measure, we develop a simple inclusion measure for ranking the interval neutrosophic sets. Finally, a multi attribute decision making problem is presented to show effectiveness of proposed inclusion measure, and results obtained are discussed. Though having a simple measure for calculation, the inclusion measure presents a new approach for handling the interval neutrosophic information.**

Index Terms— **Inclusion (subsethood) measure , Interval neutrosophic sets, Neutrosophic sets, Multi attribute decision making, Single valued neutrosophic sets**

I. I. INTRODUCTION

The concept of the neutrosophic set developed by Smarandache [12] is a set model which generalizes the the classic set, fuzzy set [21], interval fuzzy set [14] intuitionistic fuzzy set [1] and interval valued intuitionistic fuzzy set [2]. In contrast to intuitionistic fuzzy sets and also interval valued intuitionistic fuzzy sets, indeterminacy degree of an element in a universe of discourse is expressed explicitly in the neutrosophic set. There are three membership functions such that truth membership, indeterminacy membership and falsity membership in a neutrosophic set, and they are independent. However, the neutrosophic set generalizes the above mentioned sets from philosophical point of view and its functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]0^-,1^+[$ and are defined by $T_A(x): X \to]0^-,1^+[$, $I_A(x): X \to]0^-,1^+[$ and $F_A(x): X \to]0^-,1^+[$. That is, its components $T(x), I(x), F(x)$ are non-standard subsets included in the unitary nonstandard interval $]0^-,1^+[$or standard subsets included in the unitary standard interval [0, 1] as in the intuitionistic fuzzy set. Furthermore, the connectors in the intuitionistic fuzzy set are only defined by $T(x)$ and $F(x)$ (i.e. truth-membership and falsity-membership), hence the indeterminacy $I(x)$ is what is left from 1, while in the neutrosophic set, they can be defined by any of them (no restriction) [12]. However, the neutrosophic set is to be difficult to use in real scientific or engineering applications. So Wang et al. [5]-[6] defined the concepts of single valued neutrosophic set (SVNS) and interval neutrosophic set (INS) which is an instance of a neutrosophic set. At present, studies on the SVNSs and INSs are progressing rapidly in many different aspects [4]-[9]-[13]-[17]-[18]-[19]. Recently, Şahin and Küçük [13] proposed the subsethood (inclusion) measure for single valued neutrosophic sets and applied it to a multi criteria decision making problem with information of single valued neutrosophic sets.

Fuzzy entropy, distance measure and similarity measure are three basic concepts used in fuzzy sets theory. Usually subsethood measures are constructed using implication operators, t-norms or t-conorns, entropy measures or cardinalities. In classical theory, it is said that a set A is a subset of B and is denoted by $A \subset B$ if every element of A is an element of B, whenever X is a universal set and A, B are two sets in X. Therefore, inclusion measure should be two valued for crisp sets. That is, either A is precisely subset of B or vice versa. But since an element x in universal set X can belong to a fuzzy set A to varying degrees, it is notable to consider situations describing as being "more and less" a subset of another set and to measure the degree of this inclusion. Fuzzy inclusion allows a given fuzzy set to contain another to some degree between 0 and 1. According to Zadeh's fuzzy set containment, a fuzzy set B contains a fuzzy set A if $m_A(x) \le m_B(x)$, for all x in X, in which m_A and m_B are the membership functions of A and B, respectively. Various researcher have proposed different inclusion measures [3]-[7]-[8]-[10]-[11]-[15]-[16]-[20].

In this paper, we firstly review the systems of axioms of Young's fuzzy inclusion measure. Then we extend the inclusion measure of single valued neutrosophic sets to interval neutrosophic environment and give a new system of axioms for inclusion measure of interval neutrosophic sets. Moreover, we utilize the neutrosophic inclusion measure to rank the interval neutrosophic sets. To demonstrate the effectiveness of the proposed inclusion measure, we consider a multi attribute decision-making problem.

II. RELIMINARIES

In the following we give a brief review of some preliminaries.

2.1 Neutrosophic set

Definition 2.1 [12] Let X be a space of points (objects) and $x \in X$. A neutrosophic set A in X is defined by a truth-membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or real nonstandard subsets of $]0^-,1^+[$. That is $T_A(x): X \to]0^-,1^+[$, $I_A(x): X \to]0^-,1^+[$ and $F_A(x): X \to]0^-,1^+[$. There is not restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \le \sup T_A(x) \le \sup I_A(x) \le \sup F_A(x) \le 3^+$.

Definition 2.2 [12] The complement of a neutrosophic set A is denoted by A^c and is defined as $T_A^c(x) = \{1^+\} \ominus T_A(x)$, $I_A^c(x) = \{1^+\} \ominus I_A(x)$ and $F_A^c(x) = \{1^+\} \ominus F_A(x)$ for all $x \in X$.

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Definition 2.3 [12] A neutrosophic set A is contained in the other neutrosophic set B, $A \subseteq B$ iff $\inf T_A(x) \le \inf T_B(x)$, $\sup T_A(x) \le \sup T_B(x)$, $\inf I_A(x) \ge \inf I_B(x)$, $\sup I_A(x) \ge \sup I_B(x)$and $\inf F_A(x) \ge \inf F_B(x)$, $\sup F_A(x) \ge \sup F_B(x)$ for all $x \in X$.

In the following, we will adopt the representations $u_A(x)$, $p_A(x)$ and $v_A(x)$ instead of $T_A(x)$, $I_A(x)$ and $F_A(x)$, respectively.

2.2 Single valued neutrosophic sets

A single valued neutrosophic set has been defined in [5] as follows:

Definition 2.4 [5] Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form

$$A = \{\langle x, u_A(x), p_A(x), v_A(x)\rangle : x \in X\},$$

where $u_A(x): X \to [0,1]$, $p_A(x): X \to [0,1]$ and $v_A(x): X \to [0,1]$ with $0 \le u_A(x) + p_A(x) + v_A(x) \le 3$ for all $x \in X$. The values $u_A(x), p_A(x)$ and $v_A(x)$ denote the truth- membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively.

Definition 2.5 [5] The complement of a single valued neutrosophic set A is denoted by A^c and is defined as $u_A^c(x) = v(x)$, $p_A^c(x) = 1 - p_A(x)$, and $v_A^c(x) = u(x)$ for all $x \in X$. That is,

$$A^c = \{\langle x, v_A(x), 1 - p_A(x), u_A(x)\rangle : x \in X\}.$$

Definition 2.6 [5] A single valued neutrosophic set A is contained in the other single valued neutrosophic set B, $A \subseteq B$, iff $u_A(x) \le u_B(x)$, $p_A(x) \ge p_B(x)$ and $v_A(x) \ge v_B(x)$ for all $x \in X$.

Definition 2.7 ([5] Two single valued neutrosophic sets A and B are equal, written as $A = B$, iff $A \subseteq B$ and $B \subseteq A$. Moreover, we denote the family of all the single valued neutrosophic sets by $SVNS(X)$.

2.3 Interval neutrosophic sets

An interval neutrosophic set is a model of a neutrosophic set, which can be used to handle uncertainty in fieds of scientific, environment and engineering. We introduce the definition of an interval neutrosophic set as follows.

Definition 2.8 [6] Let X be a space of points (objects) and Int[0,1] be the set of all closed subsets of [0,1]. An interval neutrosophic A in X is defined with the form

$$A = \{\langle x, u_A(x), p_A(x), v_A(x)\rangle : x \in X\}$$

where $u_A(x): X \to \mathrm{int}[0,1]$, $p_A(x): X \to \mathrm{int}[0,1]$ and $v_A(x): X \to \mathrm{int}[0,1]$ with $0 \le \sup u_A(x) + \sup p_A(x) + \sup v_A(x) \le 3$ for all $x \in X$. The intervals $u_A(x), p_A(x)$ and $v_A(x)$ denote the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively.

For convenience, if let $u_A(x) = [u_A^L(x), u_A^U(x)]$, $p_A(x) = [p_A^L(x), p_A^U(x)]$ and $v_A(x) = [v_A^L(x), v_A^U(x)]$, then

$$A = \{\langle x, [u_A^L(x), u_A^U(x)], [p_A^L(x), p_A^U(x)], [v_A^L(x), v_A^U(x)]\rangle : x \in X\}$$

with the condition, $0 \le \sup u_A^U(x) + \sup p_A^U(x) + \sup v_A^U(x) \le 3$ for all $x \in X$. Here, we only take the sub-unitary interval of [0,1]. Therefore, an interval neutrosophic set is clearly neutrosophic set.

Definition 2.9 [6] Let $INS(X)$ denote the family of all the interval neutrosophic sets in universe X, assume $A, B \in INS(X)$ such that

$$A = \{\langle x, [u_A^L(x), u_A^U(x)], [p_A^L(x), p_A^U(x)], [v_A^L(x), v_A^U(x)]\rangle : x \in X\}$$
$$B = \{\langle x, [u_B^L(x), u_B^U(x)], [p_B^L(x), p_B^U(x)], [v_B^L(x), v_B^U(x)]\rangle : x \in X\}$$

then some operations can be defined as follows:

(1) $A \cup B = \{\langle x, [\max\{u_A^L(x), u_B^L(x)\}, \max\{u_A^U(x), u_B^U(x)\}], [\min\{p_A^L(x), p_B^L(x)\}, \min\{p_A^U(x), p_B^U(x)\}], [\min\{v_A^L(x), v_B^L(x)\}, \min\{v_A^U(x), v_B^U(x)\}]\rangle : x \in X\}$;

(2) $A \cap B = \{\langle x, [\min\{u_A^L(x), u_B^L(x)\}, \min\{u_A^U(x), u_B^U(x)\}], [\max\{p_A^L(x), p_B^L(x)\}, \max\{p_A^U(x), p_B^U(x)\}], [\max\{v_A^L(x), v_B^L(x)\}, \max\{v_A^U(x), v_B^U(x)\}]\rangle : x \in X\}$;

(3) $A^c = \{\langle x, [v_A^L(x), v_A^U(x)], [1 - p_A^U(x), 1 - p_A^L(x)], [u_A^L(x), u_A^U(x)]\rangle : x \in X\}$;

(4) $A \subseteq B$, if $u_A^L(x) \le u_B^L(x)$, $u_A^U(x) \le u_B^U(x)$, $p_A^L(x) \ge p_B^L(x)$, $p_A^U(x) \ge p_B^U(x)$ and $v_A^L(x) \ge v_B^L(x)$, $v_A^U(x) \ge v_B^U(x)$ for all $x \in X$.

(5) $A = B$, if $A \subseteq B$ and $B \subseteq A$.

Definition 2.10 [6] Let A be an interval neutrosophic set over X.

(1) An interval neutrosophic set over X is empty, denoted by $\tilde{A} = \underline{1}$ if $u_A(x) = [1,1]$, $p_A(x) = [0,0]$ and $v_A(x) = [0,0]$ for all $x \in X$.

(2) An interval neutrosophic set over X is absolute, denoted by $\Phi = \underline{0}$ if $u_A(x) = [0,0]$, $p_A(x) = [1,1]$ and $v_A(x) = [1,1]$ for all $x \in X$.

2.4 Subsethood measure

It is well known that subsethood measures can be generated from distance measures. In fuzzy set theory, fuzzy subsethood is an important concept. Zadeh's subsethood definition is given by for fuzzy sets A and B

$$A \subseteq B \Leftrightarrow m_A(x) \le m_B(x), \forall x \in X.$$

Since a element x in universal set X can belong to a fuzzy set A to varying degrees, it is more natural to consider an indicator of degree to which A is subset of B. In general, such an indicator is a mapping $I: FS(X) \times FS(X) \to [0,1]$ satisfying special properties, called an inclusion indicator or subsethood measure. A systems of axioms of fuzzy subsethood characterized by Young [20] is given in the following:

Definition 2.11 [20] A mapping $\alpha: FS(X) \times FS(X) \to [0,1]$ is called a fuzzy subsethood measure, if α satisfies the following properties (for all $A, B, C \in FS(X)$):

(1) $\alpha(A, B) = 1$ if and only if $A \subseteq B$.

(2) Let $\left[\frac{1}{2}\right] \subseteq A$, where $\left[\frac{1}{2}\right]$ is the fuzzy set of X defined by $m_{\left[\frac{1}{2}\right]}(x) = \frac{1}{2}$ for each $x \in X$. Then $\alpha(A, A^c) = 0$ if and only if $A = X$.

(3) If $A \subseteq B \subseteq C$, then $\alpha(C, A) \le \alpha(B, A)$; and if $A \subseteq B$, then $\alpha(C, A) \le \alpha(C, B)$.

Now, we define the cross-entropy measure of interval neutrosophic sets.

III. INCLUSION MEASURES FOR INTERVAL NEUTROSOPHIC SETS

In this section, we give a formal definition of inclusion measure for interval neutrosophic sets.

Assume that $d: INS(X) \times INS(X) \to \mathrm{R}^+ \cup \{0\}$ is a distance between interval neutrosophic sets in X. To establish the inclusion indicator expressing the degree to which A belongs to B, we use the distance between interval neutrosophic sets A and $A \cap B$. If it is considered the inclusion measure based on distance measure, we have the formal given by

$$I_d(A,B) = 1 - d(A, A \cap B)$$

In this paper, we adopt the normalized Hamming distance between interval neutrosophic sets ([18]),

$$A = \{\langle x, [u_A^L(x), u_A^U(x)], [p_A^L(x), p_A^U(x)], [v_A^L(x), v_A^U(x)]\rangle : x \in X\}$$
$$B = \{\langle x, [u_B^L(x), u_B^U(x)], [p_B^L(x), p_B^U(x)], [v_B^L(x), v_B^U(x)]\rangle : x \in X\},$$
$$d_{nH} = \frac{1}{6n}\sum_{i=1}^{n}\{|u_A^L(x_i) - u_B^L(x_i)| + |p_A^L(x_i) - p_B^L(x_i)| + |v_A^L(x_i) - v_B^L(x_i)| + |u_A^U(x_i) - u_B^U(x_i)| + |p_A^U(x_i) - p_B^U(x_i)|, |v_A^U(x_i) - v_B^U(x_i)|\}.$$

Definition 3.1 A mapping $I: INS(X) \times INS(X) \to 0,1]$ is called a inclusion measure for interval neutrosophic sets, if I satisfies the following properties (for all $A, B, C \in INS(X)$).

(1) $I(A,B) = 1$ if $A \subseteq B$.
(2) $I(A, A^c) = 1 \Leftrightarrow \forall x \in X, [u_A^L(x), u_A^U(x)] \leq [v_A^L(x), v_A^U(x)]$ and $[p_B^L(x), p_B^U(x)] \leq [0.5, 0.5]$
(3) $I(\underline{1}, \underline{0}) = 0$,
where $\underline{1}$ is the interval absolute neutrosophic set and $\underline{0}$ is the interval empty neutrosophic set.
(4) $A \subseteq B \subseteq C \Rightarrow I(C,A) \leq I(B,A)$ and $I(C,A) \leq I(C,B)$.

Theorem 3.2 Suppose that $I: INS(X) \times INS(X) \to [0,1]$ such that

$$I(A,B) = 1 - d_{nH}(A, A \cap B)$$

where d_{nH} is a normalized Hamming distance between interval neutrosophic sets, is a mapping. Then $I(A,B)$ is an inclusion measure expressing the degree to which A belongs to B.

IV. MULTI-ATTRIBUTE NEUTROSOPHIC DECISION-MAKING METHOD BASED ON THE INCLUSION MEASURE

In the following, we apply the above inclusion measure to multiattribute decision making problem based on INSs.

Example 4.1 Let us consider the following pattern recognition problem. Assume A_1, A_2, A_3 and A_4 are given four known patterns which correspond to five decision alternatives d_1, d_2, d_3 and d_4, respectively. The patterns are denoted by the following INSs in $X = \{x_1, x_2, x_3, x_4\}$.

$$A_1 = \begin{Bmatrix} \langle x_1, [0.4,0.5], [0.2,0.3], [0.3,0.4]\rangle, \langle x_2, [0.4,0.6], [0.1,0.3], [0.2,0.4]\rangle \\ \langle x_3, [0.3,0.4], [0.3,0.5], [0.4,0.5]\rangle, \langle x_4, [0.5,0.6], [0.1,0.2], [0.1,0.3]\rangle \end{Bmatrix}$$
$$A_2 = \begin{Bmatrix} \langle x_1, [0.5,0.6], [0.1,0.3], [0.2,0.3]\rangle, \langle x_2, [0.6,0.7], [0.4,0.5], [0.2,0.3]\rangle \\ \langle x_3, [0.5,0.6], [0.2,0.3], [0.3,0.4]\rangle, \langle x_4, [0.4,0.7], [0.1,0.3], [0.1,0.2]\rangle \end{Bmatrix}$$
$$A_3 = \begin{Bmatrix} \langle x_1, [0.3,0.5], [0.3,0.5], [0.3,0.4]\rangle, \langle x_2, [0.1,0.3], [0.2,0.4], [0.5,0.6]\rangle \\ \langle x_3, [0.2,0.5], [0.1,0.2], [0.4,0.5]\rangle, \langle x_4, [0.2,0.3], [0.3,0.4], [0.4,0.6]\rangle \end{Bmatrix}$$
$$A_4 = \begin{Bmatrix} \langle x_1, [0.3,0.4], [0.3,0.4], [0.3,0.4]\rangle, \langle x_2, [0.4,0.7], [0.1,0.3], [0.1,0.2]\rangle \\ \langle x_3, [0.4,0.5], [0.2,0.3], [0.3,0.4]\rangle, \langle x_4, [0.4,0.6], [0.1,0.3], [0.2,0.4]\rangle \end{Bmatrix}$$

Given an unknown sample (i.e., the positive ideal solution of decision)

$$A^+ = \begin{Bmatrix} \langle x_1, [0.5,0.6], [0.1,0.3], [0.1,0.3]\rangle, \langle x_2, [0.6,0.7], [0.1,0.3], [0.1,0.2]\rangle \\ \langle x_3, [0.5,0.6], [0.1,0.2], [0.3,0.4]\rangle, \langle x_4, [0.5,0.7], [0.1,0.2], [0.1,0.2]\rangle \end{Bmatrix}$$

Our aim is to classify pattern A^+ to one of the decision alternatives d_1, d_2, d_3 and d_4.
Using the Theorem 3.2, we can compute the inclusion measure for INSs as follows:

$$I(A^+, A_1) = 0.895,\ I(A^+, A_2) = 0.948 \text{ and},$$
$$I(A^+, A_3) = 0.779, I(A^+, A_4) = 0.916.$$

Since $I(A^+, A_2) = \max_{1 \leq i \leq 4} I(A^+, A_i)$, then the pattern A^+ should be classified to A_2 according to the principle of inclucion measure between INSs. It means that the decision alternative d_2 is the optimal alternative which is the closest alternative to positive ideal solution.

V. CONCLUSIONS

In this paper, we have suggested a set of axioms for the inclusion measure in a family of interval neutrosophic sets. Moreover, we have proposed a simple and natural inclusion measure based on the normalized Hamming distance between interval neutrosophic sets. To analyze its performance, a classification problem is established in multi attribute decision making method under interval neutrosophic environment. We hope that the findings in this paper will help the researchers to enhance and promote the further study on inclusion measure to carry out general framework for the applications in practical life.

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