

# Fourth ABC Index and Fifth GA Index of Certain Special Molecular Graphs

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**Abstract**— Several chemical indices have been introduced in theoretical chemistry to measure the properties of molecular structures, such as atom bond connectivity index and geometric-arithmetic index. In this paper, we present the fourth atom bond connectivity index and fifth geometric-arithmetic index of fan molecular graph, wheel molecular graph, gear fan molecular graph, gear wheel molecular graph, and their  $r$ -corona molecular graphs.

**Index Terms**— fourth atom bond connectivity index, fifth geometric-arithmetic index,  $r$ -corona molecular graph

## I. INTRODUCTION

Atom bond connectivity index, geometric-arithmetic index and other chemical indices are introduced to reflect certain structural features of organic molecules (See Yan et al., [1-2], Gao et al., [3-4], Gao and Shi [5], Gao and Wang [6], Xi and Gao [7-8], Xi et al., [9], Gao et al., [10] for more detail). The notation and terminology used but undefined in this paper can be found in [11].

M. Ghorbani et al., introduced the fourth Atom-Bond Connectivity index  $ABC_4(G)$  [12] as

$$ABC_4(G) = \sum_{uv \in E(G)} \sqrt{\frac{S(u) + S(v) - 2}{S(u)S(v)}}$$

where  $S(v) = \sum_{uv \in E(G)} d(u)$ . More results on fourth ABC index can refer to [13-15].

Similarity, the fifth geometric-arithmetic index was denoted by [16] as

$$GA_5(G) = \sum_{uv \in E(G)} \frac{2\sqrt{S(u)S(v)}}{S(u) + S(v)}$$

More conclusions on  $GA_5$  can refer to [17-18].

We define the general fifth geometric-arithmetic index

$$as\ GA_5^k(G) = \sum_{uv \in E(G)} \left( \frac{2\sqrt{S(u)S(v)}}{S(u) + S(v)} \right)^k$$

Let  $P_n$  and  $C_n$  be path and cycle with  $n$  vertices. The molecular graph  $F_n = \{v\} \vee P_n$  is called a fan molecular graph and the molecular graph  $W_n = \{v\} \vee C_n$  is called a wheel molecular graph. Molecular graph  $I_r(G)$  is called  $r$ -crown molecular graph of  $G$  which splicing  $r$  hang edges for every vertex in  $G$ . By adding one vertex in every two adjacent

vertices of the fan path  $P_n$  of fan molecular graph  $F_n$ , the resulting molecular graph is a subdivision molecular graph called gear fan molecular graph, denote as  $\tilde{F}_n$ . By adding one vertex in every two adjacent vertices of the wheel cycle  $C_n$  of wheel molecular graph  $W_n$ , The resulting molecular graph is a subdivision molecular graph, called gear wheel molecular graph, denoted as  $\tilde{W}_n$ .

In this paper, we present the fourth Atom-Bond Connectivity index of  $I_r(F_n)$ ,  $I_r(W_n)$ ,  $I_r(\tilde{F}_n)$  and  $I_r(\tilde{W}_n)$ . Also, the fifth geometric-arithmetic index and its general version of  $I_r(F_n)$ ,  $I_r(W_n)$ ,  $I_r(\tilde{F}_n)$  and  $I_r(\tilde{W}_n)$  are derived.

## II FOURTH ATOM-BOND CONNECTIVITY INDEX

**Theorem1.**

$$\begin{aligned} ABC_4(I_r(F_n)) &= \\ &r \sqrt{\frac{(4n-4) + r(n+2)}{(n+r)((3n-2) + r(n+1))}} + \\ &2 \sqrt{\frac{(4n-1) + r(n+4)}{((3n-2) + r(n+1))(n+3r+3)}} \\ &+ 2 \sqrt{\frac{(4n+1) + r(n+5)}{((3n-2) + r(n+1))(n+4r+5)}} + (n-4) \sqrt{\frac{(4n+2) + r(n+5)}{((3n-2) + r(n+1))(n+4r+6)}} \\ &+ 2 \sqrt{\frac{2n+8r+9}{(n+4r+5)(n+4r+6)}} + \frac{n-3}{n+4r+6} \sqrt{2n+8r+10} + \\ &2r \sqrt{\frac{n+4r+3}{(n+3r+3)(2+r)}} \\ &2r \sqrt{\frac{n+5r+6}{(n+4r+5)(3+r)}} + (n-4)r \sqrt{\frac{n+5r+7}{(n+4r+6)(3+r)}} \end{aligned}$$

**Proof.** Let  $P_n = v_1 v_2 \dots v_n$  and the  $r$  hanging vertices of  $v_i$  be  $v_i^1, v_i^2, \dots, v_i^r$  ( $1 \leq i \leq n$ ). Let  $v$  be a vertex in  $F_n$  beside  $P_n$ , and the  $r$  hanging vertices of  $v$  be  $v^1, v^2, \dots, v^r$ . By the definition of fourth atom-bond connectivity index, we have

$$\begin{aligned} ABC_4(I_r(F_n)) &= \sum_{i=1}^r \sqrt{\frac{S(v) + S(v^i) - 2}{S(v)S(v^i)}} + \\ &\sum_{i=1}^n \sqrt{\frac{S(v) + S(v_i) - 2}{S(v)S(v_i)}} + \sum_{i=1}^{n-1} \sqrt{\frac{S(v_i) + S(v_{i+1}) - 2}{S(v_i)S(v_{i+1})}} \\ &+ \sum_{i=1}^n \sum_{j=1}^r \sqrt{\frac{S(v_i) + S(v_i^j) - 2}{S(v_i)S(v_i^j)}} = \end{aligned}$$

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$$\begin{aligned}
 & r\sqrt{\frac{(4n-4)+r(n+2)}{(n+r)((3n-2)+r(n+1))}} \\
 & + 2\sqrt{\frac{(4n-1)+r(n+4)}{((3n-2)+r(n+1))(n+3r+3)}} \\
 & + 2\sqrt{\frac{(4n+1)+r(n+5)}{((3n-2)+r(n+1))(n+4r+5)}} \\
 & + (n-4)\sqrt{\frac{(4n+2)+r(n+5)}{((3n-2)+r(n+1))(n+4r+6)}} \\
 & + (2\sqrt{\frac{2n+8r+9}{(n+4r+5)(n+4r+6)}} + (n-3)\sqrt{\frac{2n+8r+10}{(n+4r+6)(n+4r+6)}}) \\
 & + (2r\sqrt{\frac{n+4r+3}{(n+3r+3)(2+r)}} \\
 & + 2r\sqrt{\frac{n+5r+6}{(n+4r+5)(3+r)}} + (n-4)r\sqrt{\frac{n+5r+7}{(n+4r+6)(3+r)}})
 \end{aligned}$$

□

**Corollary1.**

$$\begin{aligned}
 ABC_4(F_n) &= 2\sqrt{\frac{4n-1}{(3n-2)(n+3)}} \\
 & + 2\sqrt{\frac{4n+1}{(3n-2)(n+5)}} + (n-4)\sqrt{\frac{4n+2}{(3n-2)(n+6)}} \\
 & + 2\sqrt{\frac{2n+9}{(n+5)(n+6)}} + \frac{n-3}{n+6}\sqrt{2n+10}.
 \end{aligned}$$

**Theorem 2.**

$$\begin{aligned}
 ABC_4(I_r(W_n)) &= r\sqrt{\frac{r(n+2)+4n-2}{(r(n+1)+3n)(n+r)}} \\
 & + n\sqrt{\frac{r(n+5)+4n+4}{(r(n+1)+3n)(n+4r+6)}} \\
 & + \frac{n}{n+4r+6}\sqrt{2n+8r+10} + nr\sqrt{\frac{n+5r+7}{(n+4r+6)(r+3)}}.
 \end{aligned}$$

**Proof.** Let  $C_n=v_1v_2\dots v_n$  and  $v_i^1, v_i^2, \dots, v_i^r$  be the  $r$  hanging vertices of  $v_i$  ( $1 \leq i \leq n$ ). Let  $v$  be a vertex in  $W_n$  beside  $C_n$ , and  $v^1, v^2, \dots, v^r$  be the  $r$  hanging vertices of  $v$ . According to the definition of fourth atom-bond connectivity index, we get

$$\begin{aligned}
 ABC_4(I_r(W_n)) &= \sum_{i=1}^r \sqrt{\frac{S(v)+S(v^i)-2}{S(v)S(v^i)}} \\
 & + \sum_{i=1}^n \sqrt{\frac{S(v)+S(v_i)-2}{S(v)S(v_i)}} + \sum_{i=1}^n \sqrt{\frac{S(v_i)+S(v_{i+1})-2}{S(v_i)S(v_{i+1})}} \\
 & + \sum_{i=1}^n \sum_{j=1}^r \sqrt{\frac{S(v_i)+S(v_i^j)-2}{S(v_i)S(v_i^j)}} \\
 & = r\sqrt{\frac{r(n+2)+4n-2}{(r(n+1)+3n)(n+r)}}
 \end{aligned}$$

$$\begin{aligned}
 & + n\sqrt{\frac{r(n+5)+4n+4}{(r(n+1)+3n)(n+4r+6)}} \\
 & + n\sqrt{\frac{2n+8r+10}{(n+4r+6)(n+4r+6)}} + nr\sqrt{\frac{n+5r+7}{(n+4r+6)(r+3)}}.
 \end{aligned}$$

□

**Corollary2.**

$$ABC_4(W_n) = n\sqrt{\frac{4n+4}{3n(n+6)}} + \frac{n}{n+6}\sqrt{2n+10}.$$

**Theorem3.**

$$\begin{aligned}
 ABC_4(I_r(\tilde{F}_n)) &= r\sqrt{\frac{(4n-4)+r(n+2)}{(n+r)((3n-2)+r(n+1))}} \\
 & + 2\sqrt{\frac{(4n-2)+r(n+4)}{((3n-2)+r(n+1))(n+3r+2)}} \\
 & + (n-2)\sqrt{\frac{4n+r(n+5)}{((3n-2)+r(n+1))(n+4r+4)}} \\
 & + 2r\sqrt{\frac{n+4r+2}{(n+3r+2)(2+r)}} + (n-2)r\sqrt{\frac{n+5r+5}{(n+4r+4)(r+3)}} \\
 & + 2\sqrt{\frac{n+6r+5}{(n+3r+2)(3r+5)}} \\
 & + 2(n-3)\sqrt{\frac{n+7r+8}{(n+4r+4)(3r+6)}} + 2\sqrt{\frac{n+7r+7}{(n+4r+4)(3r+5)}} \\
 & + 2r\sqrt{\frac{4r+5}{(3r+5)(r+2)}} + (n-2)r\sqrt{\frac{4r+6}{(3r+6)(r+2)}}.
 \end{aligned}$$

**Proof.** Let  $P_n=v_1v_2\dots v_n$  and  $v_{i,i+1}$  be the adding vertex between  $v_i$  and  $v_{i+1}$ . Let  $v_i^1, v_i^2, \dots, v_i^r$  be the  $r$  hanging vertices of  $v_i$  ( $1 \leq i \leq n$ ). Let  $v_{i,i+1}^1, v_{i,i+1}^2, \dots, v_{i,i+1}^r$  be the  $r$  hanging vertices of  $v_{i,i+1}$  ( $1 \leq i \leq n-1$ ). Let  $v$  be a vertex in  $F_n$  beside  $P_n$ , and the  $r$  hanging vertices of  $v$  be  $v^1, v^2, \dots, v^r$ . By virtue of the definition of fourth atom-bond connectivity index, we obtain

$$\begin{aligned}
 ABC_4(I_r(\tilde{F}_n)) &= \sum_{i=1}^r \sqrt{\frac{S(v)+S(v^i)-2}{S(v)S(v^i)}} \\
 & + \sum_{i=1}^n \sqrt{\frac{S(v)+S(v_i)-2}{S(v)S(v_i)}} + \sum_{i=1}^n \sum_{j=1}^r \sqrt{\frac{S(v_i)+S(v_i^j)-2}{S(v_i)S(v_i^j)}} \\
 & + \sum_{i=1}^{n-1} \sqrt{\frac{S(v_i)+S(v_{i,i+1})-2}{S(v_i)S(v_{i,i+1})}} + \sum_{i=1}^{n-1} \sqrt{\frac{S(v_{i,i+1})+S(v_{i+1})-2}{S(v_{i,i+1})S(v_{i+1})}} \\
 & + \sum_{i=1}^{n-1} \sum_{j=1}^r \sqrt{\frac{S(v_{i,i+1})+S(v_{i,i+1}^j)-2}{S(v_{i,i+1})S(v_{i,i+1}^j)}}
 \end{aligned}$$

$$\begin{aligned}
 &= r\sqrt{\frac{(4n-4)+r(n+2)}{(n+r)((3n-2)+r(n+1))}} \\
 &+ (2\sqrt{\frac{(4n-2)+r(n+4)}{((3n-2)+r(n+1))(n+3r+2)}} \\
 &+ (n-2)\sqrt{\frac{4n+r(n+5)}{((3n-2)+r(n+1))(n+4r+4)}} \\
 &+ (2r\sqrt{\frac{n+4r+2}{(n+3r+2)(2+r)}} + (n-2)r\sqrt{\frac{n+5r+5}{(n+4r+4)(r+3)}}) \\
 &+ \left(\sqrt{\frac{n+6r+5}{(n+3r+2)(3r+5)}} \right. \\
 &+ (n-3)\left(\sqrt{\frac{n+7r+8}{(n+4r+4)(3r+6)}} + \sqrt{\frac{n+7r+7}{(n+4r+4)(3r+5)}}\right) \\
 &+ \left(\sqrt{\frac{n+6r+5}{(n+3r+2)(3r+5)}} \right. \\
 &+ (n-3)\left(\sqrt{\frac{n+7r+8}{(n+4r+4)(3r+6)}} + \sqrt{\frac{n+7r+7}{(n+4r+4)(3r+5)}}\right) \\
 &+ (2r\sqrt{\frac{4r+5}{(3r+5)(r+2)}} + (n-2)r\sqrt{\frac{4r+6}{(3r+6)(r+2)}}). \square
 \end{aligned}$$

**Corollary3.**

$$\begin{aligned}
 ABC_4(\tilde{F}_n) &= 2\sqrt{\frac{4n-2}{(3n-2)(n+2)}} \\
 &+ (n-2)\sqrt{\frac{4n}{(3n-2)(n+4)}} + 2\sqrt{\frac{n+5}{5(n+2)}} \\
 &+ 2(n-3)\sqrt{\frac{n+8}{6(n+4)}} + 2\sqrt{\frac{n+7}{5(n+4)}}.
 \end{aligned}$$

**Theorem4.**

$$\begin{aligned}
 ABC_4(I_r(\tilde{W}_n)) &= r\sqrt{\frac{r(n+2)+4n-2}{(r(n+1)+3n)(n+r)}} \\
 &+ n\sqrt{\frac{r(n+5)+4n+4}{(r(n+1)+3n)(n+4r+6)}} \\
 &+ nr\sqrt{\frac{n+5r+7}{(n+4r+6)(r+3)}} + 2n\sqrt{\frac{n+7r+10}{(n+4r+6)(3r+6)}} \\
 &+ nr\sqrt{\frac{4r+6}{(3r+6)(r+2)}}.
 \end{aligned}$$

**Proof.** Let  $C_n=v_1v_2\dots v_n$  and  $v$  be a vertex in  $W_n$  beside  $C_n$ ,  $v_{i,i+1}$  be the adding vertex between  $v_i$  and  $v_{i+1}$ . Let  $v^1, v^2, \dots, v^r$  be the  $r$  hanging vertices of  $v$  and  $v_i^1, v_i^2, \dots, v_i^r$  be the  $r$  hanging vertices of  $v_i$  ( $1 \leq i \leq n$ ). Let  $v_{n,n+1}=v_{1,n}$  and  $v_{i,i+1}^1, v_{i,i+1}^2, \dots, v_{i,i+1}^r$  be the  $r$  hanging vertices of  $v_{i,i+1}$  ( $1 \leq i \leq n$ ). In view of the definition of fourth atom-bond connectivity index, we deduce

$$\begin{aligned}
 &+ ABC_4(I_r(\tilde{W}_n)) = \sum_{i=1}^r \sqrt{\frac{S(v)+S(v^i)-2}{S(v)S(v^i)}} + \\
 &\sum_{i=1}^n \sqrt{\frac{S(v)+S(v_i)-2}{S(v)S(v_i)}} + \sum_{i=1}^n \sum_{j=1}^r \sqrt{\frac{S(v_i)+S(v_i^j)-2}{S(v_i)S(v_i^j)}} \\
 &+ \sum_{i=1}^n \sqrt{\frac{S(v_i)+S(v_{i,i+1})-2}{S(v_i)S(v_{i,i+1})}} + \sum_{i=1}^n \sqrt{\frac{S(v_{i,i+1})+S(v_{i+1})-2}{S(v_{i,i+1})S(v_{i+1})}} \\
 &+ \sum_{i=1}^n \sum_{j=1}^r \sqrt{\frac{S(v_{i,i+1})+S(v_{i,i+1}^j)-2}{S(v_{i,i+1})S(v_{i,i+1}^j)}} \\
 &= r\sqrt{\frac{r(n+2)+4n-2}{(r(n+1)+3n)(n+r)}} + \\
 &n\sqrt{\frac{r(n+5)+4n+4}{(r(n+1)+3n)(n+4r+6)}} + \\
 &nr\sqrt{\frac{n+5r+7}{(n+4r+6)(r+3)}} + n\sqrt{\frac{n+7r+10}{(n+4r+6)(3r+6)}} \\
 &+ n\sqrt{\frac{n+7r+10}{(n+4r+6)(3r+6)}} + nr\sqrt{\frac{4r+6}{(3r+6)(r+2)}}. \square
 \end{aligned}$$

**Corollary4.**  $ABC_4(\tilde{W}_n) = n\sqrt{\frac{4n+4}{3n(n+6)}}$

$$+ 2n\sqrt{\frac{n+10}{6(n+6)}}.$$

### III GENERAL FIFTH GEOMETRIC-ARITHMETIC INDEX

The terminologies for these special molecular graphs similar as Theorem 1- Theorem 4.

**Theorem5.**

$$\begin{aligned}
 GA_5^k(I_r(F_n)) &= \\
 &r\left(\frac{2\sqrt{(n+r)((3n-2)+r(n+1))}}{(4n-2)+r(n+2)}\right)^k + \\
 &2\left(\frac{2\sqrt{((3n-2)+r(n+1))(n+3r+3)}}{(4n+1)+r(n+4)}\right)^k \\
 &+ 2\left(\frac{2\sqrt{((3n-2)+r(n+1))(n+4r+5)}}{(4n+3)+r(n+5)}\right)^k + (n-4)\left(\frac{2\sqrt{((3n-2)+r(n+1))(n+4r+6)}}{(4n+4)+r(n+5)}\right)^k \\
 &+ 2\left(\frac{2\sqrt{(n+4r+5)(n+4r+6)}}{2n+8r+11}\right)^k + (n-3)\left(\frac{2\sqrt{(n+4r+6)(n+4r+6)}}{2n+8r+12}\right)^k \\
 &+ 2r\left(\frac{2\sqrt{(n+3r+3)(2+r)}}{n+4r+5}\right)^k \\
 &+ 2r\left(\frac{2\sqrt{(n+4r+5)(3+r)}}{n+5r+8}\right)^k + (n-4)r\left(\frac{2\sqrt{(n+4r+6)(3+r)}}{n+5r+9}\right)^k.
 \end{aligned}$$

**Proof.** By the definition of general fifth geometric-arithmetic index, we have

$$\begin{aligned}
 GA_5^k(I_r(F_n)) &= \sum_{i=1}^r \left( \frac{2\sqrt{S(v)S(v^i)}}{S(v)+S(v^i)} \right)^k + = r \left( \frac{2\sqrt{(r(n+1)+3n)(n+r)}}{r(n+2)+4n} \right)^k + \\
 & \sum_{i=1}^n \left( \frac{2\sqrt{S(v)S(v_i)}}{S(v)+S(v_i)} \right)^k + \sum_{i=1}^{n-1} \left( \frac{2\sqrt{S(v_i)S(v_{i+1})}}{S(v_i)+S(v_{i+1})} \right)^k + n \left( \frac{2\sqrt{(r(n+1)+3n)(n+4r+6)}}{r(n+5)+4n+6} \right)^k + \\
 & \sum_{i=1}^n \sum_{j=1}^r \left( \frac{2\sqrt{S(v_i)S(v_i^j)}}{S(v_i)+S(v_i^j)} \right)^k + n \left( \frac{2\sqrt{(n+4r+6)(n+4r+6)}}{2n+8r+12} \right)^k \\
 & = r \left( \frac{2\sqrt{(n+r)((3n-2)+r(n+1))}}{(4n-2)+r(n+2)} \right)^k + nr \left( \frac{2\sqrt{(n+4r+6)(r+3)}}{n+5r+9} \right)^k . \\
 & \left( \frac{2\sqrt{((3n-2)+r(n+1))(n+3r+3)}}{(4n+1)+r(n+4)} \right)^k \\
 & + 2 \left( \frac{2\sqrt{((3n-2)+r(n+1))(n+4r+5)}}{(4n+3)+r(n+5)} \right)^k + (n-4) \left( \frac{2\sqrt{((3n-2)+r(n+1))(n+4r+6)}}{(4n+4)+r(n+5)} \right)^k \\
 & + \left( \frac{2\sqrt{(n+4r+5)(n+4r+6)}}{2n+8r+11} \right)^k + (n-3) \left( \frac{2\sqrt{(n+4r+6)(n+4r+6)}}{2n+8r+12} \right)^k \\
 & + \left( 2r \left( \frac{2\sqrt{(n+3r+3)(2+r)}}{n+4r+5} \right) \right)^k \\
 & + 2r \left( \frac{2\sqrt{(n+4r+5)(3+r)}}{n+5r+8} \right)^k + (n-4)r \left( \frac{2\sqrt{(n+4r+6)(3+r)}}{n+5r+9} \right)^k . \\
 & \square
 \end{aligned}$$

**Corollary 5.**  $GA_5^k(F_n) =$

$$\begin{aligned}
 & 2 \left( \frac{2\sqrt{(3n-2)(n+3)}}{4n+1} \right)^k \\
 & + 2 \left( \frac{2\sqrt{(3n-2)(n+5)}}{4n+3} \right)^k + (n-4) \left( \frac{2\sqrt{(3n-2)(n+6)}}{4n+4} \right)^k \\
 & + 2 \left( \frac{2\sqrt{(n+5)(n+6)}}{2n+11} \right)^k + (n-3) \left( \frac{2\sqrt{(n+6)(n+6)}}{2n+12} \right)^k .
 \end{aligned}$$

**Theorem 6.**  $GA_5^k(I_r(W_n)) =$

$$\begin{aligned}
 & r \left( \frac{2\sqrt{(r(n+1)+3n)(n+r)}}{r(n+2)+4n} \right)^k + \\
 & n \left( \frac{2\sqrt{(r(n+1)+3n)(n+4r+6)}}{r(n+5)+4n+6} \right)^k + n + \\
 & nr \left( \frac{2\sqrt{(n+4r+6)(r+3)}}{n+5r+9} \right)^k .
 \end{aligned}$$

**Proof.** By the definition of general fifth geometric-arithmetic index, we have

$$\begin{aligned}
 GA_5^k(I_r(W_n)) &= \sum_{i=1}^r \left( \frac{2\sqrt{S(v)S(v^i)}}{S(v)+S(v^i)} \right)^k + \\
 & \sum_{i=1}^n \left( \frac{2\sqrt{S(v)S(v_i)}}{S(v)+S(v_i)} \right)^k + \sum_{i=1}^n \left( \frac{2\sqrt{S(v_i)S(v_{i+1})}}{S(v_i)+S(v_{i+1})} \right)^k + \\
 & \sum_{i=1}^n \sum_{j=1}^r \left( \frac{2\sqrt{S(v_i)S(v_i^j)}}{S(v_i)+S(v_i^j)} \right)^k
 \end{aligned}$$

**Corollary 6.**  $GA_5^k(W_n) = n \left( \frac{2\sqrt{3n(n+6)}}{4n+6} \right)^k + n.$

**Theorem 7.**  $GA_5^k(I_r(\tilde{F}_n)) =$

$$\begin{aligned}
 & r \left( \frac{2\sqrt{(n+r)((3n-2)+r(n+1))}}{(4n-2)+r(n+2)} \right)^k + \\
 & 2 \left( \frac{2\sqrt{((3n-2)+r(n+1))(n+3r+2)}}{4n+r(n+4)} \right)^k \\
 & + (n-2) \left( \frac{2\sqrt{((3n-2)+r(n+1))(n+4r+4)}}{(4n+2)+r(n+5)} \right)^k + \\
 & 2r \left( \frac{2\sqrt{(n+3r+2)(2+r)}}{n+4r+4} \right)^k \\
 & + (n-2)r \left( \frac{2\sqrt{(n+4r+4)(r+3)}}{n+5r+7} \right)^k + \\
 & \left( \frac{2\sqrt{(n+3r+2)(3r+5)}}{n+6r+7} \right)^k \\
 & + (n-3) \left( \frac{2\sqrt{(n+4r+4)(3r+6)}}{n+7r+10} \right)^k \\
 & + \left( \frac{2\sqrt{(n+4r+4)(3r+5)}}{n+7r+9} \right)^k + \\
 & \left( \frac{2\sqrt{(n+3r+2)(3r+5)}}{n+6r+7} \right)^k \\
 & + (n-3) \left( \frac{2\sqrt{(n+4r+4)(3r+6)}}{n+7r+10} \right)^k \\
 & + \left( \frac{2\sqrt{(n+4r+4)(3r+5)}}{n+7r+9} \right)^k + \\
 & 2r \left( \frac{2\sqrt{(3r+5)(r+2)}}{4r+7} \right)^k + (n-2)r \left( \frac{2\sqrt{(3r+6)(r+2)}}{4r+8} \right)^k .
 \end{aligned}$$

**Proof.** By virtue of the definition of general fifth geometric-arithmetic index, we get

$$\begin{aligned}
 GA_5^k(I_r(\tilde{F}_n)) &= \sum_{i=1}^r \left( \frac{2\sqrt{S(v)S(v^i)}}{S(v)+S(v^i)} \right)^k + \\
 & \sum_{i=1}^n \left( \frac{2\sqrt{S(v)S(v_i)}}{S(v)+S(v_i)} \right)^k + \sum_{i=1}^n \sum_{j=1}^r \left( \frac{2\sqrt{S(v_i)S(v_i^j)}}{S(v_i)+S(v_i^j)} \right)^k +
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{i=1}^{n-1} \left( \frac{2\sqrt{S(v_i)S(v_{i+1})}}{S(v_i) + S(v_{i+1})} \right)^k + \sum_{i=1}^{n-1} \left( \frac{2\sqrt{S(v_{i+1})S(v_i)}}{S(v_{i+1}) + S(v_i)} \right)^k + 2n \left( \frac{2\sqrt{(n+4r+6)(3r+6)}}{n+7r+12} \right)^k + \\
 & \sum_{i=1}^{n-1} \sum_{j=1}^r \left( \frac{2\sqrt{S(v_{i,i+1})S(v_{i,i+1}^j)}}{S(v_{i,i+1}) + S(v_{i,i+1}^j)} \right)^k + nr \left( \frac{2\sqrt{(3r+6)(r+2)}}{4r+8} \right)^k. \\
 = & r \left( \frac{2\sqrt{(n+r)((3n-2)+r(n+1))}}{(4n-2)+r(n+2)} \right)^k + GA_5^k(I_r(\tilde{W}_n)) = \sum_{i=1}^r \left( \frac{2\sqrt{S(v)S(v^i)}}{S(v) + S(v^i)} \right)^k + \\
 & \sum_{i=1}^n \left( \frac{2\sqrt{S(v)S(v_i)}}{S(v) + S(v_i)} \right)^k + \sum_{i=1}^n \sum_{j=1}^r \left( \frac{2\sqrt{S(v_i)S(v_i^j)}}{S(v_i) + S(v_i^j)} \right)^k + \\
 & \sum_{i=1}^n \left( \frac{2\sqrt{S(v_i)S(v_{i,i+1})}}{S(v_i) + S(v_{i,i+1})} \right)^k + \sum_{i=1}^n \left( \frac{2\sqrt{S(v_{i,i+1})S(v_{i,i+1})}}{S(v_{i,i+1}) + S(v_{i,i+1})} \right)^k + \\
 & \sum_{i=1}^n \sum_{j=1}^r \left( \frac{2\sqrt{S(v_{i,i+1})S(v_{i,i+1}^j)}}{S(v_{i,i+1}) + S(v_{i,i+1}^j)} \right)^k = r \left( \frac{2\sqrt{(r(n+1)+3n)(n+r)}}{r(n+2)+4n} \right)^k + \\
 & n \left( \frac{2\sqrt{(r(n+1)+3n)(n+4r+6)}}{r(n+5)+4n+6} \right)^k + nr \left( \frac{2\sqrt{(n+4r+6)(r+3)}}{n+5r+9} \right)^k + \\
 & n \left( \frac{2\sqrt{(n+4r+6)(3r+6)}}{n+7r+12} \right)^k + n \left( \frac{2\sqrt{(n+4r+6)(3r+6)}}{n+7r+12} \right)^k + \\
 & nr \left( \frac{2\sqrt{(3r+6)(r+2)}}{4r+8} \right)^k. \square
 \end{aligned}$$

**Corollary7.**  $GA_5^k(\tilde{F}_n) = 2 \left( \frac{2\sqrt{(3n-2)(n+2)}}{4n} \right)^k + (n-2) \left( \frac{2\sqrt{(3n-2)(n+4)}}{4n+2} \right)^k + \left( \frac{2\sqrt{5(n+2)}}{n+7} \right)^k + 2(n-3) \left( \frac{2\sqrt{6(n+4)}}{n+10} \right)^k + 2 \left( \frac{2\sqrt{5(n+4)}}{n+9} \right)^k + \left( \frac{2\sqrt{5(n+2)}}{n+7} \right)^k.$

**Theorem8.**  $GA_5^k(I_r(\tilde{W}_n)) = r \left( \frac{2\sqrt{(r(n+1)+3n)(n+r)}}{r(n+2)+4n} \right)^k + n \left( \frac{2\sqrt{(r(n+1)+3n)(n+4r+6)}}{r(n+5)+4n+6} \right)^k + nr \left( \frac{2\sqrt{(n+4r+6)(r+3)}}{n+5r+9} \right)^k$

**Corollary8.**  $GA_5^k(\tilde{W}_n) = n \left( \frac{2\sqrt{3n(n+6)}}{4n+6} \right)^k + 2n \left( \frac{2\sqrt{6(n+6)}}{n+12} \right)^k.$

IV FIFTH GEOMETRIC-ARITHMETIC INDEX

By taking  $k=1$  in Theorem 5-8, we get the following conclusions on fifth geometric-arithmetic index.

**Theorem9.**  $GA_5(I_r(F_n)) = \frac{2r\sqrt{(n+r)((3n-2)+r(n+1))}}{(4n-2)+r(n+2)} + \frac{4\sqrt{((3n-2)+r(n+1))(n+3r+3)}}{(4n+1)+r(n+4)} + \frac{4\sqrt{((3n-2)+r(n+1))(n+4r+5)}}{(4n+3)+r(n+5)} + \frac{2(n-4)\sqrt{((3n-2)+r(n+1))(n+4r+6)}}{(4n+4)+r(n+5)} + \frac{4\sqrt{(n+4r+5)(n+4r+6)}}{2n+8r+11} + \frac{2(n-3)\sqrt{(n+4r+6)(n+4r+6)}}{2n+8r+12}$

$$\frac{4r\sqrt{(n+3r+3)(2+r)}}{n+4r+5} + \frac{4r\sqrt{(n+4r+5)(3+r)}}{n+5r+8} + \frac{2(n-4)r\sqrt{(n+4r+6)(3+r)}}{n+5r+9}$$

Corollary9.

$$GA_5(F_n) = \frac{4\sqrt{(3n-2)(n+3)}}{4n+1} + \frac{4\sqrt{(3n-2)(n+5)}}{4n+3} + \frac{2(n-4)\sqrt{(3n-2)(n+6)}}{4n+4} + \frac{4\sqrt{(n+5)(n+6)}}{2n+11} + \frac{2(n-3)(n+6)}{2n+12}$$

Theorem

$$GA_5(I_r(W_n)) = \frac{2r\sqrt{(r(n+1)+3n)(n+r)}}{r(n+2)+4n} + \frac{2n\sqrt{(r(n+1)+3n)(n+4r+6)}}{r(n+5)+4n+6} + n + \frac{2nr\sqrt{(n+4r+6)(r+3)}}{n+5r+9}$$

Corollary 10.  $GA_5(W_n) = \frac{2n\sqrt{3n(n+6)}}{4n+6} + n$

Theorem

$$GA_5(I_r(\tilde{F}_n)) = \frac{2r\sqrt{(n+r)((3n-2)+r(n+1))}}{(4n-2)+r(n+2)} + \frac{4\sqrt{((3n-2)+r(n+1))(n+3r+2)}}{4n+r(n+4)} + \frac{2(n-2)\sqrt{((3n-2)+r(n+1))(n+4r+4)}}{(4n+2)+r(n+5)} + \frac{4r\sqrt{(n+3r+2)(2+r)}}{n+4r+4} + \frac{2(n-2)r\sqrt{(n+4r+4)(r+3)}}{n+5r+7} + \frac{2\sqrt{(n+3r+2)(3r+5)}}{n+6r+7} + \frac{2(n-3)\sqrt{(n+4r+4)(3r+6)}}{n+7r+10} + \frac{2\sqrt{(n+4r+4)(3r+5)}}{n+7r+9} + \frac{2\sqrt{(n+3r+2)(3r+5)}}{n+6r+7} + \frac{2(n-3)\sqrt{(n+4r+4)(3r+6)}}{n+7r+10} + \frac{2\sqrt{(n+4r+4)(3r+5)}}{n+7r+9} + \frac{4r\sqrt{(3r+5)(r+2)}}{4r+7} + \frac{2(n-2)r\sqrt{(3r+6)(r+2)}}{4r+8}$$

Corollary11.

$$GA_5(\tilde{F}_n) = \frac{\sqrt{(3n-2)(n+2)}}{n} + \frac{(n-2)\sqrt{(3n-2)(n+4)}}{2n+1}$$

$$\frac{2\sqrt{5(n+2)}}{n+7} + \frac{2(n-3)\sqrt{6(n+4)}}{n+10} + \frac{2\sqrt{5(n+4)}}{n+9} + \frac{2\sqrt{5(n+2)}}{n+7} + \frac{2(n-3)\sqrt{6(n+4)}}{n+10} + \frac{2\sqrt{5(n+4)}}{n+9}$$

Theorem12.

$$GA_5(I_r(\tilde{W}_n)) = \frac{2r\sqrt{(r(n+1)+3n)(n+r)}}{r(n+2)+4n} + \frac{2n\sqrt{(r(n+1)+3n)(n+4r+6)}}{r(n+5)+4n+6} + \frac{2nr\sqrt{(n+4r+6)(r+3)}}{n+5r+9} + \frac{4n\sqrt{(n+4r+6)(3r+6)}}{n+7r+12} + \frac{2nr\sqrt{(3r+6)(r+2)}}{4r+8}$$

Corollary 12.  $GA_5(\tilde{W}_n) = \frac{2n\sqrt{3n(n+6)}}{4n+6} + \frac{2n\sqrt{6(n+6)}}{n+12} + \frac{2n\sqrt{6(n+6)}}{n+12}$

10.

11.

### V. CONCLUSION

In this paper, we determine the fourth atom bond connectivity index and fifth geometric-arithmetic index of fan molecular graph, wheel molecular graph, gear fan molecular graph, gear wheel molecular graph, and their  $r$ -corona molecular graphs. Furthermore, the general version of fifth geometric-arithmetic index is discussed.

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